Supporting Students To Develop Mathematical Explanation:
Studying the Work of Teaching

by

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DEDICATION

To my family

ACKNOWLEDGEMENTS

My twenty-year journey in mathematics education, beginning as an undergraduate, cannot be fully described here. Finishing this long journey as a student, many unforgettable moments rush into my mind including my first attendance to the ICME conference (ICME10) in Copenhagen, when my academic advisor and co-chair, Hyman Bass, gave a presidential plenary lecture, while I did not even dare to imagine that I can get his invaluable guidance and thoughtful comments in person at the University of Michigan but just had a vague dream of pursing a doctorate degree. Walking through those memories, I would like to express my deepest gratitude to mentors, teachers, colleagues, friends, and family who provided tremendous support that made it possible for my dream to come true.

The dilemma I had as an elementary school teacher lead me to study classroom discourse, classroom interactions, talk, language, communication, and whole-group discussion for many years, but my dissertation ideas are largely shaped by invaluable guidance from my dissertation committee. First of all, I am deeply grateful to Hyman Bass who made me to be able to see instructional interactions from a "disciplinary perspective" and expatiated its underlying ideas that I cannot get from just reading articles. His disciplinary rich but pedagogically insightful comments, unusual but sophisticated language choices, perspicacious remarks, and conceptual inquiries about the rationale for my initial ideas transformed my dissertation in a very significant way. I have been privileged to have Deborah Ball, who provided invaluable guidance in the formulation of my research questions and the design of study, addressed the core issues that I did not even notice before, and generously gave me permission to analyze this

fascinating data set. My dissertation ideas are fundamentally built on and informed by her work about studying teaching. I thank Kevin Miller, who provided rich resources around explanations and challenged me to elaborate the conceptualization of mathematical explanation. No words are enough to express my deepest gratitude to my mentor, supervisor, and co-chair Mark Hoover. In every step of my graduate life, I am deeply indebted to him who provided incredible intellectual scaffolding, trained many important academic and practical skills, guided me to engage in what is entailed in doing research, challenged me to make arguments and to build rhetorical structure beyond describing research findings, patiently encouraged me to stay focused on writing a dissertation, continuously assured me how to navigate a giant maze of a dissertation without feeling lost but enjoying the process and ultimately help me find the exit of the maze with a feeling of accomplishment. Many research projects that he supervised created great opportunities for me to learn about mathematical knowledge for teaching (MKT) and the work of teaching. My dissertation committee triggered my intellectual curiosity, guided disciplined ways of doing research, provided analytical scaffolding, made it possible for me to see the fruition of a dissertation, and furnished important skills and knowledge that I can harness for my long-term academic career.

During this journey, I also have been fortunate to have amazing teachers and students who have transformed my life in many different ways. My 10th grader teacher, Ms. Guisuk Park, who is my role model as a teacher, influenced me on becoming a teacher, built my confidence, and shaped my identities during high school. For over twenty years, she has been a great supporter to me in pursuing my dream. My undergraduate professor, Professor Seung-In Nam, provided very useful resources to study teaching and learning mathematics beyond undergraduate coursework—otherwise such resources were very limitedly available in Korea in the late 1990s. He also made me dream of a bigger world, motivated me to pursue graduate school, cultivated my potential, and created many opportunities both in research and teaching. My graduate advisor, Professor JeongSuk Pang, brought me to the international conference in my first year of my master's program, which made me dream of pursuing a doctorate abroad. Besides providing rich scholarly resources in research on teaching from the master's program, she continuously encouraged me through emails and calls during the journey. I

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Finishing this journey, I am excited to see what stories will fill the next chapters of my life.

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ABSTRACT

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The practice of explanation is widely seen as key to developing students' ability to understand mathematics, but it has not been well cultivated in mathematics classrooms. Efforts have been made to increase students' discursive activities but these have largely overlooked the specific ways of constructing knowledge in the discipline. Approaching the practice of explanation on disciplinary grounds, this dissertation conceptualizes the work of teaching entailed in supporting students to develop mathematical explanation by decomposing the work into its constituent components.

To provide an empirical basis for such a conceptualization, I analyzed instructional interactions managed by one teacher, teaching the same mathematical tasks to five different cohorts of students sampled from the same school district. Designing the study to hold other variables of instruction (teacher and content) relatively constant while only varying the class composition of students, this study leverages a core predicament of teaching—its dependence on students—to explore how instructional interactions unfold

with different groups of students and how instruction might be adjusted to meet different students' needs. By further performing this analysis with multiple mathematical tasks, this dissertation identifies "mathematical-task generality" and "mathematical-task specificity" in constructing mathematical explanations and elucidates how mathematical tasks shape the level of mathematical supports and the role of instructional resources.

This dissertation decomposes the work of teaching entailed in supporting students to develop mathematical explanation into four core tasks of teaching and two instructional resources, while articulating a structure and rationale for such a decomposition. Beyond the benefits of specifying this important aspect of teaching, this study serves to address one targeted problem of teaching—the uncertainties and unexpectedness arising with students—by gradually increasing the demands of unpacking students' ideas and the complexities of managing multiple interactions among students so that teaching is rendered more understandable, doable, and learnable by teachers. This dissertation advances a study design that addresses the problem of how the particularities of students matter for teaching, contributes to establishing a grammar of decomposition by specifying the decomposition's underlying rationale, and provides conceptual tools that can be used to develop effective teacher education.

CHAPTER 1.
THE RESEARCH PROBLEM

1.1. Overview

Chapter 1 begins with examining why the practice of explanation, as key to develop mathematical power and proficiency, plays an important role as a crucial vehicle for learning, a social construct in discipline, a pivotal moment in teaching, and a researchable moment of teaching. It then discusses challenges in cultivating the practice of explanation and reviews studies that use proxies to approximate the richness of mathematical explanation. After a brief examination of a vignette to provide a glimpse of characteristics of students' initial explanation and endemic challenges of supporting students to develop mathematical explanation, this chapter ends by introducing the focus of this dissertation, the central premise that shapes this dissertation, and the organization of this dissertation.

1.2. The Practice of Explanation Is Important

There is widespread agreement that the ability for all students to explain their mathematical thinking is key to developing mathematical power and proficiency. The *Principles and Standards for School Mathematics* (National Council of Teachers of Mathematics, 2000) propose that all students should be able to organize mathematical thinking through communication, communicate mathematical thinking coherently and clearly to others, analyze and evaluate the mathematical thinking of others, and use the language of mathematics precisely. In a similar vein, the National Research Council's report, *Adding It Up* (National Research Council, 2001), highlights the need for adaptive reasoning, including informal explanation, justification, and intuitive and inductive reasoning, as well as formal proof and deductive reasoning. Consistent with the national calls for students to be able to engage in mathematical explanation, the *Common Core State Standards* similarly propose that mathematically proficient students should be able to construct viable arguments, critique others' reasoning, and communicate precisely to others.

Besides the shared goals set up for students to achieve in school mathematics across these documents, the practice of explanation plays an important role as a crucial vehicle for learning, a social construct in the discipline, a pivotal moment in teaching, and a researchable moment of teaching. The detailed arguments about why the practice of explanation is important in these domains are discussed below.

1.2.1. The Practice of Explanation as a Crucial Vehicle for Learning

The practice of explanation has been considered an important goal for learning because giving, hearing, and evaluating explanations facilitate resolution of cognitive dissonance, cognitive development, and active participation in the process of knowledge construction. More specifically, in an effort to develop skills to make one's explanation understandable for others, giving an explanation can serve as an opportunity for students to reflect on their own thinking and to reconstruct and better articulate their existing knowledge.

For instance, examining the relationship between spontaneous self-explanation and problem solving skills in physics, Chi, Bassok, Lewis, Reimann, and Glaser (1989) showed that more successful students produced a greater number of spontaneous self-explanation than less successful students. Extending this finding, Chi, De Leeuw, Chiu, and LaVancher (1994) examined the effect of prompting explanations in learning declarative knowledge and demonstrated that students who were prompted to generate self-explanation after reading texts had better achievement gains than students who merely read the same texts twice but were not prompted to generate self-explanation.

Using the term "self-explanation effect," Chi and her colleagues affirmed that generating an explanation facilitates the process of integrating new information into prior knowledge. In a meta-analysis of 17 empirical studies on small-group interactions in mathematics, Webb (1991) showed that giving elaborated explanations had positive effects on mathematics achievement, but giving only an answer, a non-elaborated explanation, a procedural explanation, or a managerial explanation was not statistically significant for mathematics achievement.

At the same time, hearing others' explanations provides opportunities for students to appropriate mathematical language that a teacher or more advanced students use, to recognize any cognitive dissonance that contradicts to their own understanding, and to use others' explanations as a prompt to extend their own knowledge. For instance, in examining the effect of hearing explanations, Perry (2000) analyzed the quantity and quality of explanation in Japanese, Taiwanese, and U.S. mathematics classrooms and showed that Asian students received more frequent and generalizable explanations from their teachers than U.S. students. Unlike the effects of giving explanations, Webb (1991) showed that receiving explanations had either a marginal effect or no statistical significance on mathematics achievement, and receiving only an answer had a negative effect on mathematics achievement. Given the lack of statistical significance on the effect of receiving explanations, Webb (1991) assumed that students in the reviewed studies might not have received effective explanations, which led to difficulties in understanding, internalizing, and using the received explanations. All together, these studies provide empirical evidence that students who are exposed to rich mathematical explanation have better mathematical understanding than students who are rarely exposed to mathematical explanation.

1.2.2. The Practice of Explanation as a Social Construct in Discipline

The practice of explanation is a social construct in the discipline, with other mathematical practices such as mathematical justification, argumentation, and proof. Even though a clear distinction has not been made between these mathematical practices, the practice of explanation has been viewed as a construct that is accepted, established, and shared by the community rather than one individual's intellectual property. More detailed discussion will be provided in Chapter 2.3.

1.2.3. The Practice of Explanation as a Pivotal Moment in Teaching

Besides a crucial vehicle for learning and a social construct in the discipline, the practice of explanation is also a pivotal moment in teaching. This section elaborates its rationale into two reasons: (1) portraying the most complex work in managing dynamic instructional interactions and (2) being situated in specific contexts.

First, supporting students to develop mathematical explanations is one of the most complex domains in the work of teaching because it requires the continuous management of the dynamic interactions between a teacher and students around content during instruction (Cohen, Raudenbush, & Ball, 2003; Lampert, 2001; Leinhardt, 2001). Other domains, such as choosing examples or choosing representations, also demand the adjustment of content to be comprehensible for students and the translation of student knowledge into the legitimate form of knowledge in mathematics (e.g., Ball, 1993; R. Cohen, 2005) but the management of instructional interactions in these domains involves fewer dynamics, less uncertainty, and less instantaneity to the ideas that students propose in a public space on the spot than supporting students to develop mathematical explanation. Because of its sensitivity to language, one ambiguous word chosen by an initial student could be inaccurately taken up by another student, which turns into pandemonium in managing the dynamic interactions and proliferates the complexities of mathematical issues to be resolved.

Second, supporting students to develop mathematical explanation is situated in specific contexts (Ball & Forzani, 2010; Leinhardt, 2001). Even if a teacher uses the same mathematical task, the initial explanation that a student proposes, the language that a student chooses, the sequence of explanations that several students produce, and when the key idea emerges vary from one cohort to another. In addition, the instructional contexts would be different from one case to another. More specifically, the supports needed for developing mathematical explanation at the beginning of an academic year might be quite different from the supports needed at the end of an academic year because of the established and shared norms associated with the practice of explanation.

1.2.4. The Practice of Explanation as a Researchable Moment of Teaching

Some aspects of teaching are not quite visible for observers without documenting supplementary data other than the videotaped lessons. For example, the motivations, desires, or goals that lead a practitioner to make a particular instructional decision are often invisible[1] for observers. This sometimes makes the study of teaching ambiguous and leaves much space for debate rather than arriving at agreed-on ideas. Without extensive, exhaustive, and comprehensive evidence drawn from a practitioner, examining the relationship between a particular decision and its drive seems to be equivocal. Moreover, the drives for a particular decision inferred by observers might not even be noticed by or consistent with the practitioner who actually made the decision.

Using the term "researchable teaching moment," Leinhardt (2001) articulates the specific elements of what counts as researchable teaching moment. She writes:

> The researchable teaching moment should be commonly recognizable as being a legitimate part of the instructional landscape; it should be a generally agreed-upon critical aspect of the teaching repertoire—that is, doing it well should matter; it should involve all the major actors in the drama of the class—teachers, content, and students; and it should have the potential to be reflective of differences among subject matter areas, reflective of responsiveness to the unique features of a given student group, and reflective of differences in teaching approaches. It should be a commonplace of the instructional landscape. (Leinhardt, 2001, p.338)

The practice of explanation satisfies most of the above-mentioned criteria for a researchable teaching moment. First, the practice of explanation—whether it is successfully done or poorly done—is commonly recognizable within the instructional landscape by observers without drawing substantive data from the practitioner. Second, as discussed in Chapter 1.2.1, giving and hearing rich explanations are significantly associated with students' positive achievement gains. Third, as discussed in Chapter 1.2.3, the practice of explanation involves the main actors of instruction— teacher,

[1] Instead of Lortie's notion of "invisiblity," Lewis (2007) conceptualizes teaching as invisible work in the sense that the crucial features of teaching are difficult to notice, attend, articulate, and name by observers, both by preservice teachers and inservice teachers. Instead of Lortie's notion of invisiblitiy from learners' side or Lewis's notion of invisibility as the needs to name, articulate, and decompose practices, the invisiblity in this section section refers to the idea that the drives or the desires that shape a particular decision made by the practitioner are invisible for others who observe that practices.

content, and students—and is responsive to the particular groups of students. Lastly, the practice of explanation reflects the epistemology of the discipline of mathematics, thus the practice of explanation in mathematics is quite different from the practice of explanation in history.

1.3. Cultivating The Practice of Explanation is Challenging

Now, I turn my attention to challenges in cultivating the practice of mathematical explanation. Despite its widely accepted values and increasingly pressing needs, the practice of explanation has not been well cultivated in the U.S. mathematics classrooms as evidenced by both quantitative and qualitative measures. This section discusses such challenges in cultivating the practice of mathematical explanation.

One reason might be that teachers have too much control of power, authority, and agency while restricting students' active participation to build explanations in mathematics classrooms. Or, many teachers might believe that giving explanations to students are more efficient, effective, and accurate than hearing students' explanations. Even if teachers ask students to explain, many teachers might consider it as part of a routine after presenting a solution and might not pay attention to the particulars of students' explanations. On the other hand, students often feel anxious about giving a public speech and taking risks publicly in front of the whole class. Thus, it is challenging for students to alleviate psychological anxieties and to minimize social risks.

Besides authoritative, psychological, and motivational issues, many teachers have difficulties with cultivating the practice of explanation because the process of extending knowledge is challenging in itself. Cohen (2011) asserts that teachers often present knowledge to students in a compressed, polished, and finished form rather than helping students to construct their own knowledge, even though the latter cultivates a practice of teaching that finally leads students to cultivate intellectual inquiry:

> Teachers and learners face the same gulfs of ignorance, but from different sides. Learners must somehow build bridges across the gulf, but these bridges are often fragile because the learners work from relative ignorance. The teacher's assignment is to help learners build those bridges, but they work from greater knowledge. … Rather than helping learners construct and reconstruct bridges of their own, teachers present the finished result of their learning. That reduces the

> likelihood that teachers can cultivate a practice of teaching, for it can limit learners' understanding. (p.106)

Similarly, a teacher and students face each other across the same gulfs of ignorance in building explanations. On the one side, students do not have sufficient language to explain their mathematical ideas (Forman & Larreamendy-Joerns, 1998) and their explanations are distant from disciplinary explanation (Leinhardt, 2001). On the other side, teachers often present the compressed, polished, and finished form of mathematical explanations to students rather than helping students construct their own explanations. It has not been paid enough attention until recently, but this implies that supporting students to develop mathematical explanation demands substantial mathematical knowledge for teachers to sensitize, interpret, translate, and scale up student explanation. If teachers have insufficient, inaccurate, and inappropriate mathematical knowledge, it is not feasible to unpack the mathematical meaning behind student explanation.

1.4. Using Proxies To Approximate the Practice of Explanation

Before moving into the specific problems that motivate this dissertation study, one important issue that needs to be considered is to figure out whether indicators used to approximate the richness of mathematical explanation sufficiently capture the nature of the discipline of mathematics. Given that the practice of explanation is often equated with the act of verbalizing thoughts, it would be worthwhile to examine whether some quantitative or qualitative proxies would be good candidates to predict the richness of mathematical explanation. This section examines two of such proxies: (1) the opportunity of talk and (2) the frequency, duration, and type of explanation.

As a part of the *Trends in International Mathematics and Science Study* (TIMMSS) 1999 video study, Hiebert (2003) analyzes the opportunities of talk in eighth grade mathematics classrooms in six countries[2]: Australia, Czech Republic, Hong Kong SAR, Japan, Netherlands, and U.S. Using the videotaped lessons and their corresponding translated transcripts, Hiebert (2003) analyzes the opportunity of talk in term of the following three indicators: (1) the total number of words spoken by teachers and students

[2] This book initially analyzes the TIMSS video study from seven countries, but misses one country in analyzing opportunity of talk.

during public interaction (see Figure 1.1); (2) the ratio of teacher talk to student talk (see Figure 1.2); and (3) the length of each utterance[3] by teachers and students during public interaction (see Figure 1.3 and Figure 1.4). Given that Hong Kong SAR eighth graders and Japanese eighth graders outperformed in mathematics in the TIMSS 1999 study[4], it is worthwhile to examine whether or not the data from these three indicators from U.S. differ from those of their counterparts from Hong Kong SAR and Japan.

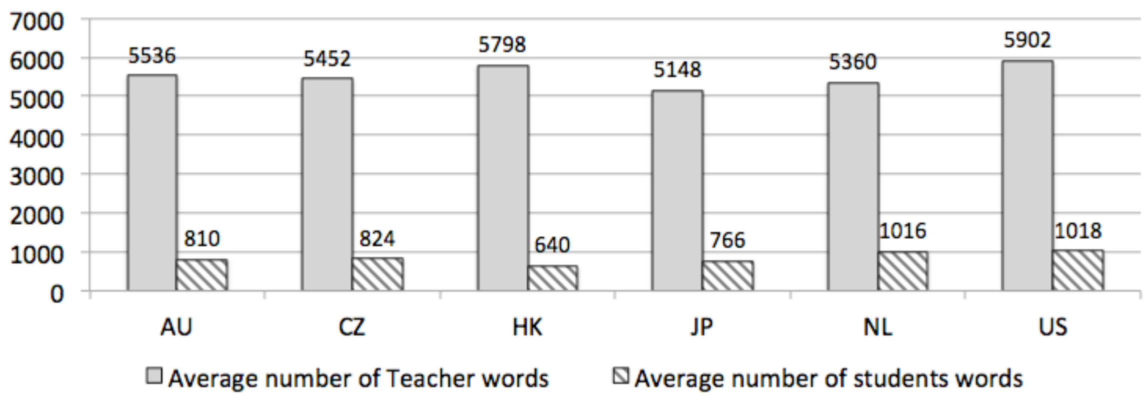

Figure 1.1. The number of words spoken by teachers and students (Hiebert, 2003, p.109)

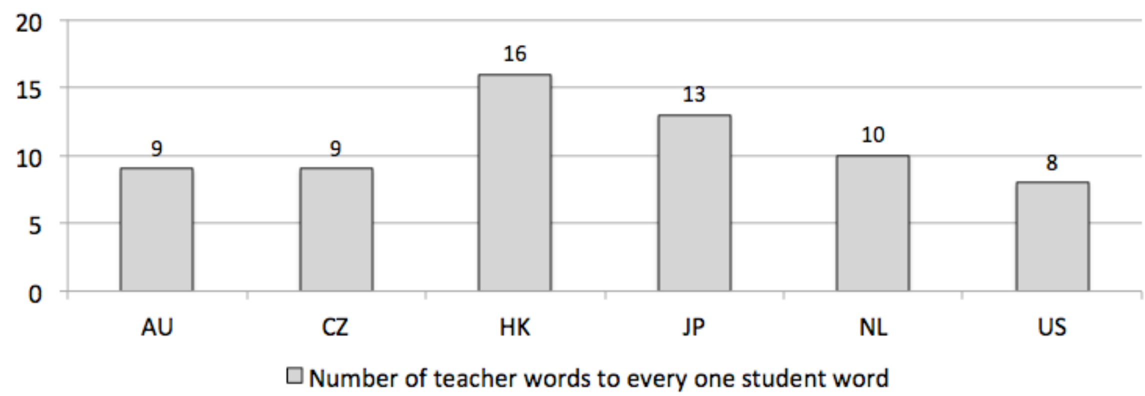

Figure 1.2. The ratio of teacher talk to student talk (Hiebert, 2003, p. 110)

[3] In the TIMSS 1999 video study, the utterance is defined as "talk by one speaker uninterrupted by another speaker." (Hiebert, 2003, p.110)
[4] Among 34 participating countires in the TIMSS 1999 study, Hong Kong SAR eighth graders ranked 4th (average score of 582), Japanese eight graders ranked 5th (average score of 579), and U.S. eight graders ranked 19th (average score of 502). Both the average score of Hong Kong SAR eighth graders and the average score of Japanese eight graders are significantly higher than the average score of U.S. eigth graders. (National Cetner for Education Statistics, 2000)

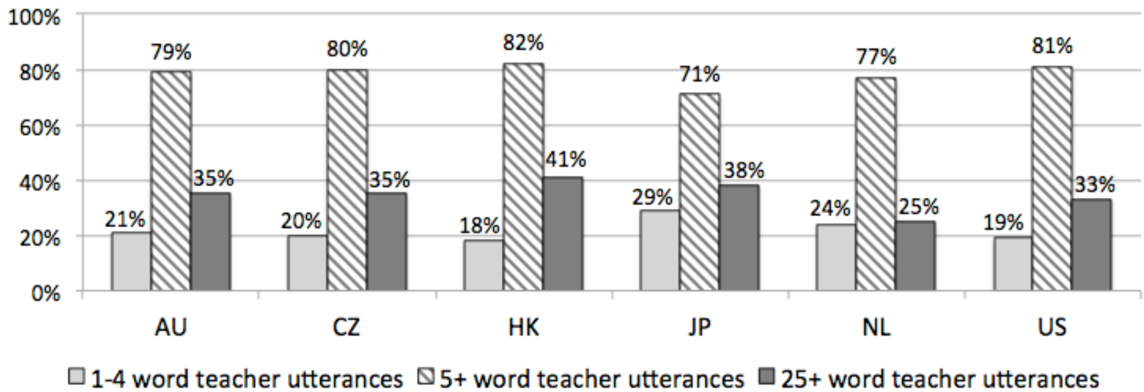

Figure 1.3. The length of each utterance by teachers (Hiebert, 2003, p. 111)

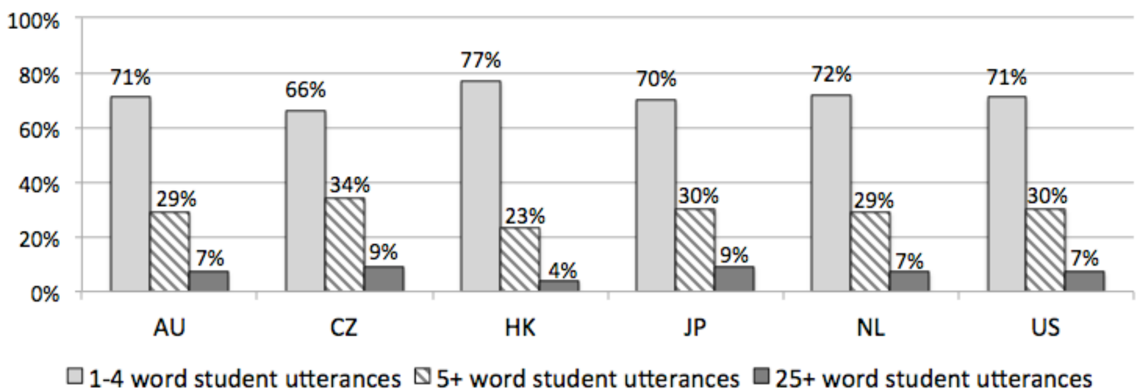

Figure 1.4. The length of each utterance by students (Hiebert, 2003, p.112)

In the first indicator, the number of words spoken by U.S. teachers was the highest (5902). It is similar to that of Hong Kong SAR teachers (5798), but is greater than that of Japanese teachers (5148). Similarly, the number of words spoken by U.S. students was also the highest (1018) among six countries, but it is noticeably greater than that of Hong Kong SAR students (640) and Japanese students (766). For the second indicator, the ratio of teacher talk to student talk in U.S. (8:1) is much smaller than the ratio of teacher talk to student talk in Hong Kong SAR (16:1) and Japan (13:1). In the last indicator, the percentage of shorter utterances (less than 5 words) by U.S. teachers and the percentage of longer utterances (more than 5 words but less than 25 words) by U.S. teachers are quite similar to those of Hong Kong SAR teachers. In addition, the percentage of shorter utterances (less than 5 words) by U.S. students and the percentage of longer utterances (more tan 5 words but less than 25 words) by U.S. students are quite similar to those of Japanese students.

These descriptive statistics might provide an approximate correlation between opportunity to talk and students' performance in mathematics, but lack strong evidence to make such causal inferences. In some sense, the results are somewhat counterintuitive to our basic assumption that the more students talk, the more they learn mathematics. Given our shared value of student talk, it is not easy to explain why U.S. eighth graders did not perform better than their counterparts from Hong Kong SAR and Japan, in spite of the fact that U.S. students talked more than their counterparts and the ratio of teacher talk to student talk was lower than other countries. As Hiebert (2003) addresses, no broad consensus about how the composition of teacher talk and student talk affect student learning has been made yet. He writes:

> Some argue that limited student talk reduces learning opportunities to those excessively weighted toward low-level skills and factually oriented instruction (Bunyi 1997; Cazden 1988; Knapp and Shields 1990). Advocates of student talk also suggest that student interaction increases the opportunities for students to elaborate, clarify, and reorganize their own thinking (Ball 1993; Hatano 1988). Others argue that student learning is best fostered by explicit or direct teaching, such as stating an objective and providing step-by-step instruction—which necessarily awards teachers substantially more talk opportunities than students (e.g., Gage 1978; Rosenshine and Stevens 1986; Walberg 1990). A third view suggests that the optimum ratio of teacher to student talk is a function of the content students are to learn (Goldenberg 1992/1993). (Hiebert, 2003, pp.107-108)

Even though the proponents of student talk are drawn from the more recent literature published after the 1990s and the proponents of teacher talk are drawn from the literature published before the 1990s, it is not easy to arrive at the incontrovertible conclusion that the effect of amount of talk on students' mathematical performance with these data. In interpreting these data, the following issues need to be considered.

One issue might be that these cross-cultural comparisons do not take account of general linguistic differences across countries. For example, even for delivering the same message in general, it is likely that the average number of words spoken in English might be longer than the average number of words spoken in Chinese or Japanese (e.g., differences in the use of auxiliary verbs; the use of definite and indefinite articles; and the omission of subjects). Even if these general linguistic differences are adjusted in the process of translation, it is also possible that the culturally shared criteria, norms, and politeness in one country might be different from another country. Thus, the simple comparison of the amount of talk might not isolate the general linguistic and cultural differences across countries.

Second, it is likely that simply counting the amount of talk might not serve well as a good predictor to explain the quality of mathematical explanation, which is the main target of this dissertation study. This does not mean that the amount of talk is not a good indicator for other studies. The amount of talk functions well for other studies which focus on the participation structure, discursive activities, authority, and identities. Given that this dissertation focuses on the development of mathematical explanation, the construct of being measured (i.e., the amount of talk) is much broader than the outcome that I intend to measure (i.e., the quality of mathematical explanation). The amount of talk measured might also include off-task talk or off-topic talk, which rather distracts, delays, and obstructs the development of mathematical explanation.

The practice of explanation has also been examined in a more refined domain than just as discursive activities. Viewing that mathematical contents are communicated through a discursive form of explanation, Perry (2000) analyzed the frequency, duration, and type of explanation in the Japanese, Taiwanese, and U.S. first-grade and fifth-grade mathematics classrooms.

As an on-site observation, the observers were instructed to make narrative notes about the flow of instruction, the descriptions of lesson, the verbal remarks made by teacher and students, and the time segments as much as possible. Each one-minute interval segment was coded into one of the nine instructional activities: (1) question-and-answer; (2) seatwork; (3) question-and-answer with seatwork; (4) explanation; (5) evaluation; (6) choral responses; (7) teacher gives directions; (8) mental calculation; and (9) other. If an instructional segment includes "explanations of how to do something and/or of why to so something" (p.185), it was coded as explanation. The exposure to these explanations was further categorized into two levels: (1) if the explanation lasted at least one minute, it was coded as extended explanation and (2) if the explanation provided less than one minute, it was coded as brief explanation which was embedded in other instructional activities.

Figure 1.5 illustrates the percentage of extended explanation and three brief explanations[5] provided in first-grade mathematics classrooms in Japan, Taiwan, and U.S. The extended explanations, lasting at least one minute, were more frequently provided in Japan than Taiwan and the U.S. Perry (2000) showed that there were no significant differences in the duration of extended explanation across countries (5.91 minutes in Japan, 5.5 minutes in Taiwan, and 5.03 minutes in the U.S.). Considering that the extended explanations were more frequently provided in Japan, however, she argued that Japanese first-graders had more benefits from hearing explanations. Among three brief explanations, it is noticeable to see that the proportion of explanation embedded in question-and-answer and the proportion of explanation embedded in evaluation in Japanese and Taiwanese first-grade mathematics classroom is much greater than that of in U.S. first-grade mathematics classroom.

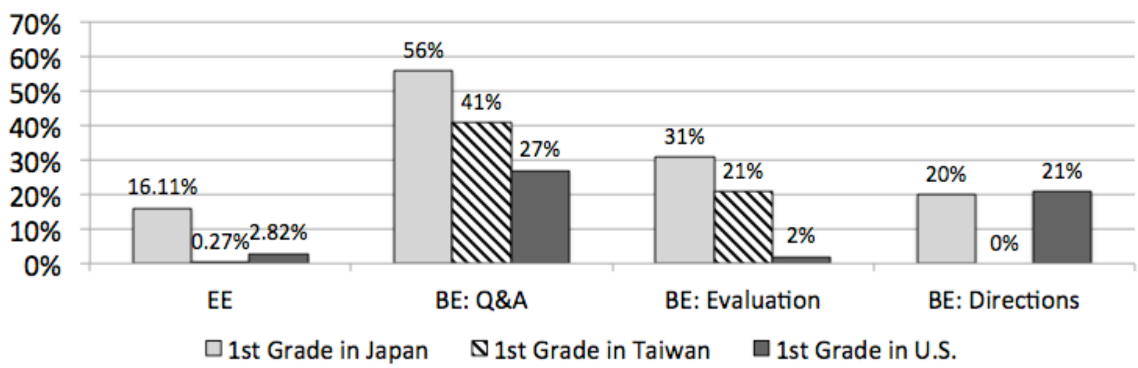

Figure 1.5. The percentage of extended explanation (EE) and brief explanation (BE) in the first grade mathematics classrooms (Perry, 2000, extracted, edited, and created a diagram from the data)

In analyzing the type of explanation about solving addition and subtraction problems from first-grade classrooms, Perry (2000) made comments that most of the explanations

[5] Among eight other instructional activities, I only included three other instructional activities (question and answer; evaluation; and teacher gives directions). First, as the author acknowledged, the explanation provided during seatwork might not be accessible to the observers in making the narrative observational note. The other two instructional activities (choral responses; mental calculation) were excluded because no segments were coded in some countries. In the first-grade mathematics classrooms, no mental calculation was observed in Japan and U.S. No choral response and mental calculation were observed in the U.S. fifth-grade mathematics classrooms.

were procedural and the conceptual explanations were rare in all of the three countries. However, she notes:

> What we learn from these examples is that the Chinese and Japanese children most often were presented with what seemed to be consistent, straightforward, and useful methods for solving addition and subtraction problems. In comparison, the U.S. children were often presented with ways of handling these same problems that were either difficult to understand (e.g., the families of explanation) or difficult to generalize to more difficult problems (e.g., it would be difficult to count on your fingers to solve multiplication problems or even to subtract 37 from 82). (p.196)

Figure 1.6 illustrates the percentage of extended explanations and three brief explanations provided in the fifth-grade mathematics classrooms in Japan, Taiwan, and U.S. In the fifth grade, the percentage of extended explanations, lasting at least one minute, was quite similar between Japan and Taiwan, which are substantially more frequently offered than in the U.S. In contrast to the first-grade mathematics classrooms, the proportion of explanation embedded in question-and-answer is quite similar across three countries. Having a similar pattern with the first-grade mathematics classrooms, however, the proportion of explanation embedded in evaluation in Japanese and Taiwanese fifth-grade classroom is much greater than inn the U.S. fifth grade classrooms.

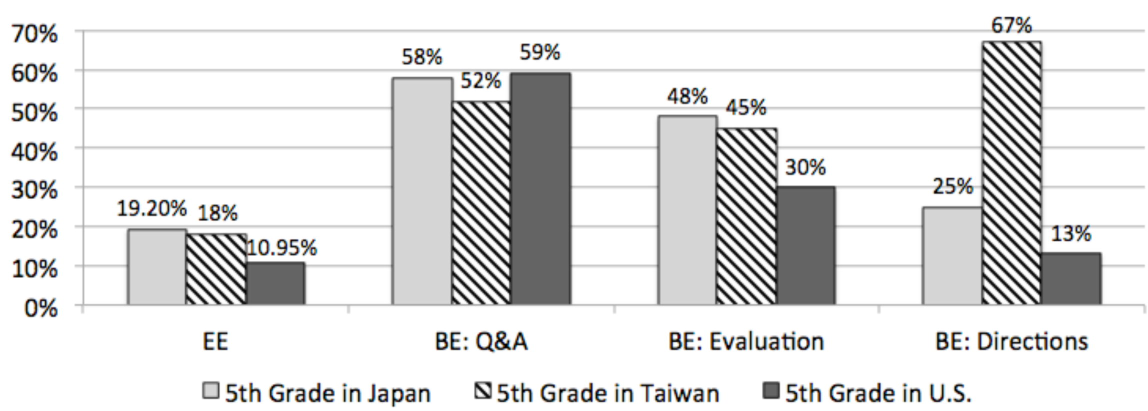

Figure 1.6. The percentage of extended explanation (EE) and brief explanation (BE) in the fifth-grade mathematics classrooms (Perry, 2000, extracted, edited, and created a diagram from the data)

In analyzing the type of explanation about fractions from the fifth-grade classrooms, Perry (2000) notes:

> Most of the Japanese explanations reminded children that fractions are composed of component parts and that the relation between these parts is important to consider. The explanation provided in the Chinese classrooms were more eclectic: Concepts were explained by pointing out alternative solutions, describing component parts, and providing examples. In the United States, it was rarely pointed out that there might be more than one way to solve a problem; instead many explanations consisted of rules or directives that lacked a description of why the rules were important. (p.200)

In analyzing the frequency, duration, and type of explanation, Perry (2000) narrows the investigation of discursive activity into how the mathematical concepts are introduced, delivered, and communicated. Nevertheless, the concept of the practice of explanation is still much broader than the concept of mathematical explanation conceptualized in this dissertation. Additionally, given that Perry coded how the explanation is delivered, offered, and provided by teachers rather than students, as well as including procedures or rule-of-thumbs, it is not clear how the students construct mathematical explanations and how teachers scale up the explanations provided by students.

Considering different grade-levels, constructs being measured, and coding procedures between Hiebert's study and Perry's study, the results from these proxies to approximate the practice of explanation seem to be mixed. In synthesizing these descriptive statistics, however, it is obvious that students do not have sufficient opportunities to learn mathematics if they are deprived of opportunities to give, hear, and evaluate explanation. However, the inverse of this statement is not always true. The take-away points from these studies can be thought of as missing disciplinary components. The practice of explanation has been often equated with discursive activities and it does not exclude other forms of knowledge that are not legitimate in the discipline.

1.5. Insufficient Disciplinary Grounds in Studying Mathematical Explanation

Cultivating the practice of explanation in mathematics classrooms has been a great interest for the last few decades, but a large body of literature mostly focuses on constructive, pedagogical, cognitive, and discursive perspectives. Given the deficiency of explanations in mathematics classrooms, a number of researchers have investigated ways of eliciting explanations in mathematics classroom. A discursive perspective suggests the use of discourse moves such as repeating, revoicing, requesting, and agreeing or disagreeing. The cognitive approach suggests the use of more cognitively demanding mathematical tasks. The sociocultural approach suggests cultivating norms. As a pioneer of cultivating norms in line with school mathematics rather than traditional mathematics, Yackel and Cobb (1996) develop the ideas of social norms and sociomathematical norms. The pedagogical approaches suggest employing strategies or techniques. Fravillig (2001) proposes three pedagogical strategies to advance children's thinking: (1) eliciting; (2) supporting; and (3) extending. Stein, Engle, Smith, and Hughes (2009) propose five pedagogical strategies to orchestrate a whole-group discussion effectively: (1) anticipating; (2) monitoring; (3) selecting; (4) sequencing; and (5) connecting.

These approaches have made great contributions to enrich the cognitive, discursive, and pedagogical opportunities, but questions remain about how to support students to develop the skills of constructing and articulating mathematical explanations that portrays a way of constructing knowledge in a more disciplinarily profound way. Ball and Bass (2000) address the lack of disciplinary perspective in studying teaching. They write:

> This mathematical perspective makes visible some critical aspects of mathematics teaching and learning that are hidden when viewed from a cognitive or sociocultural perspective. In particular, this analysis allows for and explores a subject-specific view of learning." (Ball & Bass, p.195)

One of problems that motivate this dissertation study is that the practice of explanation has been often equated with other discursive activities in general, without paying sufficient attention to profound, rigorous, and legitimate form of knowledge in the discipline.

1.6. A Vignette of Explaining "Even" And "Odd" Numbers

Before jumping into the overview of this dissertation study, this section examines one of the vignettes from Deborah Ball's third-grade public school classroom during the 1989-1990 school year[6]. The class consisted of 22 students who were racially, culturally, socioeconomically, and linguistically diverse, with varying mathematical performance at the entry to the third grade. During the one-hour long class period, the teacher, Ms. Ball, usually began the class by having students work on 'the problem of the day' individually and then having them confer with others in a small group, followed by a whole-group discussion for the rest half of the class period. The purpose of introducing this vignette is to provide a glimpse of characteristics of students' initial explanations and endemic challenges to support students to develop mathematical explanation.

The analysis of this section draws on episodes from a series of lessons about even and odd numbers taught in the middle of January 1990, when the class had been working on identifying even and odd numbers, defining the concepts of even and odd numbers, and proving whether conjectures about the properties of even and odd numbers are always true[7]. On January 16, the teacher revisited the conjectures that the students produced in the previous lesson, elicited an example for each conjecture, and then asked the students to experiment with one of the conjectures to see whether the conjecture, only tested for one example so far, always works. Each student, either alone or with a partner, experimented with one of the following four conjectures: (1) odd number + odd number = even number; (2) odd number – odd number = even number; (3) even number + even number + … + even number = even number; and (4) odd number of odd number = odd number; and even number of odd number = even number.

Near the end of the class period, the teacher observed that there was a lot of disagreement about what constitutes an even number or an odd number. Being attentive

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[6] This is a part of the large research project that documented the extensive record of practices from Deborah Ball's third-grade classroom and Magdalene Lampert's fifth-grade classroom in the public elementary school. For the overview of this record of practices, see Lampert and Ball (1998) and Lampert (2001). For the detailed description of this third-grade mathematics classroom taught by Ms. Ball, see Ball (1997).

[7] An in-depth analysis about this vignette is also provided by Ball (1997), Ball and Bass (2003), Stylianies and Ball (2008), Ball, Lewis, and Thames (2008), and Thames (2009). The main arguments vary by the published articles and by the authors to some extent.

to the lack of a “base of public knowledge” (Ball & Bass, 2003), the teacher delayed the discussion about these mathematical conjectures and guided the discussion to establish an agreed-on definition of an even number and an odd number. Prompted by the teacher’s request for defining an even or odd number, Sheena first proposed a definition of an even number.

Sheena:	I'd say that the definition for an even number is um, a number that you can split.
Student:	Yeah.
Ball:	Split what?
Sheena:	Split *<u>evenly</u>*. You split *<u>in half</u>*.
Ball:	Can you give an example? Everyone listen closely. See if you agree or disagree with her.
Sheena:	Say you have 6, so I'll make this... and then you want to have it so you can *<u>split it in half</u>* and so you split what you have, *<u>the same amount of numbers</u>* on each side.

The initial definition proposed by Sheena approximated to the key idea of divisibility by 2 but did not specify the remainder (i.e., no remainder) and the set of quotient (i.e., integers) yet. After Sheena proposed the definition of an even number, the teacher elicited questions or comments on Sheena’s proposal. Betsy wanted to propose a different definition, but the teacher stayed with Sheena’s proposal and asked Betsy whether she had any question for Sheena. As Betsy requested a clarification of what was written on the board, Sheena provided an explanation about why 6 is an even number.

Sheena:	Okay, say you have 6 ... 1, 2, 3, 4, 5, 6 and you want to have it so you can *<u>split it in half</u>* with *<u>the same numbers</u>* on each side, so you split it like this, and then you have *<u>the same amount</u>* on each side. It's even. So that's why they call it even numbers. Because you have 1, 2, 3 and 1, 2, 3.
Ball:	Lucy?
Lucy:	You can do that with odd numbers.
Student:	You can do that with odd numbers too.
Ball:	Do you want to go up and show her? Sheena, don't erase it because people do want to look at it.
Student:	If you had 3, you can go like that too.
Students:	Yes! Yeah! Yeah!
Student:	I agree.
Student:	Yeah, one and a half on each side, one and a half on each side.
Ball:	One and a half on each side? Mei?

Mei: But, I, I don't think they ... yeah, but I still don't think that you could, that you could do it. Well I think what Sheena, I think Sheena should revise to that, even numbers that have the, numbers that you can split that have *the same amount* on each side *without having to have halfs*.

In explaining how her proposed definition applies to her example, Sheena partitioned six circles into two equal groups, which could be algebraically translated into $2k$ rather than $k2$, but did not specify that there was no remainder. As shown in her language choices, such as "the same amount of numbers," "the same numbers," or "the same amount," she did not identify the set of k either. After Sheena's explanation, Lucy challenged Sheena's proposed definition and Mei suggested revising Sheena's proposed definition. Although Mei's statement is not mathematically sophisticated yet (e.g., "halfs" instead of "halves"; "without having to have halfs" instead of "no remainder"), she restricted the set of possible quotients to whole numbers. Because of the time constraint, the teacher wrapped up the discussion at this point.

On the next day, January 17, the class revisited the arguments about the definition of an even number from the previous day. Instead of asking the original authors to present their arguments, the teacher asked other students to restate the arguments about the definition of an even number. Jeannie restated Sheena's initial definition as "an even number is something that you can put, put it up *so both sides of it have the same amount*." and Keith restated Lucy's challenge as "you can *split an odd number*." Betsy restated Mei's suggestion for revision as "Sheena should say that you *have to put whole ones* on each side." and Mei affirmed that "Sheena should revise so that even number can be split having *the same amount* on each side and *not having halfs*." Following these restatements, Sheena revised her initial definition to "an even number has *whole numbers* on each side, *not half* on each side" by incorporating Betsy's restatement and Mei's restatement. Owing to Lucy's challenge with 3—it can be split equally but has one and a half on each side—, Sheena was able to further articulate that the quotient is restricted to the set of whole numbers, but excludes the set of rational numbers that are not whole numbers, such as a half. Until this moment, the class seemed to implicitly agree that 3 is an odd number because it violates the freshly developed definition of an even number in a set model representation. As the teacher asked for comments on Sheena's revised definition, Tory made a claim that 3 is an even number using a number line representation.

Tory:	Because, yesterday, um, we were, I was having with me, Riba and Ofala, we were having the same problem and we started with the 1 and on 3 we ended up with an even number.
Ball:	Could you show us?
Tory:	We started with 1 even, odd...
Students:	Odd, odd!
Student:	1 is odd!
Student:	1 is odd, you're supposed to start with the 2.
Student:	If you're going even you start with the 2's, if you're going with the 1 you go odd.
Student:	1 isn't odd.
Mei:	Zero is odd because you have nothing on each side. 1 is half, you have to cut …
Student:	1 isn't the um, isn't an odd number because um, because you could take two things and make it into one, because two halfs make it into 1.

Unlike Sheena who made a direct proof that 6 is an even number in a set model representation, Tory used mathematical induction by proving her "base case" of 1 and using the agreed-on induction rule of even-odd alternativeness in a number line model. However, her premise, the "base case" of 1, is incorrectly reasoned, so she arrived at the incorrect conclusion that 3 is an even number. Maybe being confused between "split in half" as a divisor of 2 and "split into half" as a quotient, Tory might think that 1 is an even number because 1 can be split in half. Tory's comment evoked a debate about whether 1 can be split in half or not. The opponents who refuted Tory's claim argued that 1 cannot be split in half, whereas the proponents who supported Tory's claim argued that 1 can be split in half. As Sean further argued that 1 is an even number and 2 is an even number, several students expressed disagreement with Sean because two even numbers cannot be in a row based on the even-odd alternative rule. Based on Sean's successive argument that an even number is represented as the addition of two of the same number, it could be assumed that he thought the converse of one of the previously examined conjectures as being true, which is not always true (i.e., the statement "if even number + even number = even number, even number is always expressed as even number + even number" is not true). Not reaching an agreement about whether 1 is an even number or odd number yet, Betsy made another claim that 0 is an even number because 1 is an odd number using the even-odd alternativeness. Acknowledging that 1 can be split, but Lindiwe argued that 1 is an odd number.

> Lindiwe: […] Like if you have one cookie and you split it in half, they can have half, but what Sean was saying that um, 1 is an even number, you can't um, like, um, when Sean was saying that um, 1 is an even number, I, I, I mean like, I said how, where are you getting from, like, because the numbers I said, where are you getting the other things from, and I disagree because everybody, like my mom, she told me that, um, that 1 was an odd number and 2 was an even number because you can count by two's and you could still have 1. You can give 1 to each person.

Lindiwe attempted to refute Sean's argument, but ended up with the status-based argument (Yackel & Cobb, 1996) by relying on the adult's authority after having a lot of repetition and hesitation. In addition, it is not clear how the last part of his explanation (i.e., "you can count by two's and you could still have 1. You can give 1 to each person.") supports his argument that 1 is an odd number. Sean refuted this argument again.

> Sean: It's, it's even because um, um, because if it's odd, like um, if it's odd um, how, well… one's an even number because um, because… when I was talking with my mom about even and odd's and stuff, she told me that 1 was an even number because, 1 and 2 are even numbers, because, because um, she, because when she was in school, she learned that 1 was an even number because there were, you have two halves, you put them together and make 1.

At first Sean made several attempts to do the mathematical proof by contradiction but did not have a success with that. Having difficulties with convincing why 1 is an even number, he might change his strategy by showing that if 1 is an odd number, a logical contradiction would occur. Like Lindiwe, Sean ended up with the status-based argument (Yackel & Cobb, 1996) by relying on the adult's authority. His last explanation could be algebraically translated into $k+k$, rather than $2k$, without the explicit statement that the two parts are the same amount. Despite the fact that $k+k$ and $2k$ are algebraically equivalent, the addition representation might not serve equally well to define an even number as the multiplication representation does. Sean insisted that 1 is an even number, but Cassandra challenged his argument with 0. As Sean responded that 0 is not a number, the debate whether 0 is an even number or an odd number began to arise. Using an observation about the disagreement between Jeannie and Betsy on the previous day, the teacher shifted the attention to the argument about whether 0 is an even number or not.

Jeannie: I think it's even and odd because um… I'm not really sure.
Ball: How were you, how were you trying to convince her of that yesterday?
Jeannie: Um…
Betsy: She never convinced me.
Ball: How she was trying to convince you though.
Betsy: No she didn't.
Ball: Jeannie, what were you trying to say to her?
Jeannie: Um, when zero is here, I think she thought that um, zero was even and odd.
Ball: How do you know that? What did she do that made you think that?
Jeannie: (Pause).
Ball: Did she say something one day about it?
Jeannie: No.
Ball: Why did you think she thought that?
Jeannie: I'm not sure. I want to think about it.
Ball: Okay.
Betsy: I think it's even because I know that one's odd and I've asked my parents and they said that zero is even because if you had zero of nothing, then you could cut zero in two and have zero of something on both sides.
Ball: Mei?
Betsy: Okay, this is what I'm saying, if you have zero things, zero things, and you cut it in two, you have zero stuff on both sides.

Hearing the exchange between Jeannie and Betsy, one student chimed in by saying "Half! It's half." and another student said "Yeah, you have air on the other side of it." The teacher then elicited comments from other students and Sean got the turn to share his comment.

Sean: Uh huh, I just, I don't, I disagree. Well, I *agree and disagree* because um, because I think it's true that you can cut zero in half and there'd be zero on both sides, but that would, there, but I don't, I still don't, I don't think that um, that um, you can ever, *you can*

> *ever have zero to begin with*. Because, like, what um, like, like, how, how would you get zero then? Lucy said that um, if you, if you, if you had um, if you had one below zero plus one, she said it would be zero, but I disagree and said it would be um, it would be 1 above zero, and it is.

Sean's explanation is both pedagogically and mathematically challenging to decipher. First of all, without the explicit reference about whom he agrees with or whom he disagrees with, it is not clear whether Sean considered 0 as an even number, an odd number, both, or neither. Because he insisted that 0 is nothing (and not a number) in responding to Cassandra's challenge in the previous exchange, he might think that 0 is neither an even number nor an odd number. On the next day, January 18, during the working meeting with the fourth graders, Sean disagreed with the idea that 0 is an even number by explaining "I disagree because like if… if, like zero can't be an even number or anything because you don't have two of anything to make it in the first place. Like 2 of something that's an even number." From these claims, it could be inferred that he partially agreed with partitioning 0 in a set model representation but might not be persuaded with starting from 0 in a number line model or with algebraically representing 0 as $k+k$ where k is not the same as the number (0) to be decomposed. As Sean argued that 1 is an even number because 1/2 + 1/2 = 1 in the previous exchange, he might think that 0 should be also represented with the addition of two numbers other than 0 if 0 is an even number. To support his claim, he refuted Lucy's argument -1 + 1 = 0 to prove that 0 cannot be expressed with the addition of two (same) numbers. Maybe the term "on each side" in the set model representation shared by the class was misinterpreted by Sean as "the number located on the left side of 0" (-1) and "the number located on the right side of 0" (1) in a number line representation. The class had further discussion about whether zero is even number or odd number in the working meeting with the fourth graders on January 18 and in the review of the working meeting on January 19.

Obviously, the exchanges made in this third-grade classroom are quite different from the typical U.S. mathematics classrooms. The students were highly motivated, actively participated, produced longer utterances, and were not afraid of taking risks by expressing their ideas in a public space. In addition, they were engaging in the substantial mathematical work of naming and using names, making conjectures, and evaluating conjectures (for more details about this mathematical work, see Ball, Lewis, and Thames, 2008). The teacher neither prescribed the ready-made definition of an even number nor exerted her authority to have students accept the most desirable, efficient, or accurate definition of an even number. Even in the contexts in which challenges in sharing authorities and responsibilities of explaining and challenges in alleviating psychological anxieties and minimizing social risks are resolved, the endemic challenges in bridging the gulf between teacher knowledge and student knowledge still remain.

In analyzing students' individual proposals of mathematical explanations, several general observations can be made. First, students often bring up mathematical intuitions in a public space, but they are not readily equipped with rationale to articulate their intuitions or with evidence to strengthen their intuitions. In the above excerpt, Jeannie proposed that 0 is both even and odd numbers but failed to provide the specific reasons to support her claim. Second, students often rely on authorities to show the validity of their mathematical claims. For example, Lindiwe, Sean, and Betsy referred to what their parents told them about even numbers and odd numbers to show the validity of their claims about 1 and 0. This is what Yackel and Cobb (1996) call for a mathematical basis for explanations rather than status-based as the preliminary step for developing an understanding what constitutes an acceptable mathematical explanation. Third, students often have issues of repetition, stammering, unsettled logic, and lack of reference in explaining their thinking.

Beyond theses general issues of students' initial explanations, the challenges in supporting students' development of mathematical explanations are attributable to the mathematical language, structure, and logic that students initially used in their explanations. First, the incorrect premise leads to incorrect conclusion. For example, Tory's claim is built on the premise that 1 is an even number, which is mathematically incorrectly reasoning, thus her conclusion that 3 is an even number is mathematically incorrect as well. Second, the same mathematical terms are used without paying much attention to their accurate, precise, and sophisticated mathematical meanings. For example, the term "half" was interchangeably used without making a distinction between "half" as a divisor of 2 and "half" as a quotient. In another example, the term "on each side" might be interpreted differently between a set model representation and in a number line representation. Third, the students draw on different representational models to explain the definition of an even number. Given the various representational models used, however, the mapping between representations is not successfully made by the students and the limitation of each particular representation is not explicitly noticed by the students. The set model representation makes it difficult to define an even number for zero or negative numbers (correspondence between verbal explanations and pictorial representations) and creates confusion between "split in half" as an action (as a divisor) and as a product (i.e., quotient). Fourth, the same mathematical observation is differently interpreted by and differently used for mathematical claims. For example, "zero has nothing on each side" is both used as evidence to support the claim that "zero is an even number" and the claim that "zero is an odd number." In another example, "one can be split into half" is used as evidence to support the claim that 1 is an even number and "one cannot be split into half" is used as an evidence to support the claim that 1 is an odd number. Lastly, the lack of shared references and consensus on the base of public mathematical knowledge at the beginning of a whole-group discussion makes it difficult to advance the discussion. Different reference points make it difficult to advance the discussion. In making claims about whether 1 is an even number or an odd number, some students make a reference to 2 as the starting point of even numbers, whereas others make a reference to 0 as the starting point of even numbers.

Even in contexts with rich student talk, there are endemic challenges in supporting students to develop mathematical explanation. In the following section, I introduce the focus of this dissertation, the central premise that shaped this dissertation study, and the organization of this dissertation study.

1.7. Overview of the Dissertation

1.7.1. Focus of the Dissertation

The purpose of this dissertation study is to conceptualize the work of teaching entailed in supporting students to develop mathematical explanations, and particularly, the ways of using the two key instructional resources of discourse resources and collective resources to that end. More specifically, the research questions that guide this dissertation study are:

1. *What are the core tasks of teaching to support students to develop mathematical explanation?*
2. *How are instructional resources used to support these core tasks of teaching?*
    a. *How are discourse resources used to support these core tasks of teaching?*
    b. *How are collective resources used to support these core tasks of teaching?*

This dissertation study analyzes instructional interactions managed by the same teacher teaching the same mathematical tasks to different cohorts of students, sampled from the same school district with the same recruitment procedure. Because of this sampling procedure over years, it is assumed that there are no substantial differences in students' demographic characteristics and their mathematical abilities across cohorts. Given that one of the greatest predicaments of teaching is its dependence on students (Cohen, 2011), this method for studying teaching untangles the ways in which the same teacher adjusts instruction to meet the students' need in developing mathematical explanation wherein each cohort of students bring different mathematical ideas, stance, issues, language, ambiguity, and difficulties in explaining the same mathematical task.

To provide an empirical basis for an analytical-conceptual method, I analyzed the data from the Elementary Mathematical Laboratory (EML), a two-week summer mathematics program for entering fifth graders taught by Professor Deborah Ball at the University of Michigan's School of Education, across multiple years. Among the plethora of records of practices of the EML data across five years (EML 2007, EML 2008, EML 2009, EML 2010, and EML 2013), I purposefully selected four mathematical tasks that highlight different features of mathematical explanation and have been used across multiple years: (1) the brown rectangle problem; (2) the blue and green rectangle problem; (3) the two-coin problem; and (4) the three-combination problem.

1.7.2. Teaching as Managing Dynamic Instructional Interactions: Foundation for a Theoretical, Methodological, Analytical, and Conceptual Framework

A central premise of this dissertation study is that teaching involves managing dynamic instructional interactions between a teacher and students around the content (Cohen et al., 2003). This premise has permeated into the theoretical, methodological, analytical, and conceptual frameworks of this dissertation study. In studying teaching, the alignment between theoretical, methodological, analytical, and conceptual frameworks has not been well made. It is not difficult to find such a study, for example, in which the theory is grounded in situated learning by Lave and Wenger (1991), the methodology is adopted from grounded theory by Corbin and Strauss (2008), and the analytical approach is built on discourse analysis by Gee (2014), whereas the conceptual framework solely addresses pedagogical strategies or techniques that particularly matters for teaching. Admittedly the multiple perspectives borrowed from various disciplines, such as sociology, anthropology, cognitive psychology, and linguistic, have enriched the investigation of teaching and learning mathematics. However, they often do not resolve the endemic issues or challenges that are entailed in teaching. As Hiebert and Grouws (2007) argue that theories of teaching need to be further developed in ways "in which the key components of teaching fit together to form an interactive, dynamic system for achieving particular learning goals" (p.373), a similar effort needs to be made with the alignment among theoretical, methodological, analytical, and conceptual framework that are situated in the key components of teaching. This dissertation attempts to make such an alignment in studying teaching.

The theoretical framework is grounded in the idea that teaching is managing multiple relationships simultaneously between a teacher and students around content (Cohen, Raudenbush, & Ball, 2003; Lampert, 2001; Cohen, 2013). In managing these multiple relationships, Ball and Bass (Ball, 1993; Ball & Bass, 2000) call for developing ways to intertwine bifocal perspectives—mathematical and pedagogical perspectives—

that serve the twin imperatives of responsibility to the content and responsiveness to students.

In a line with the theoretical perspective, the analytical framework draws on the idea of coordinating mathematical and pedagogical perspectives in analyzing instructional interactions (Thames, 2009). More specifically, I first paid singular attention to each perspective and then scaled up how these two perspectives complement, reinforce, and scaffold each other in supporting students to develop mathematical explanation.

Unlike an experimental study that controls variables that are not primary interests of investigation but have some effects on outcome, an observational study often does not have such control of key variables. A single case study by a single teacher teaching a single cohort of students at one time point has the potential to provide rich information and details of instructional interactions, but the validity of such findings could be threatened by other extraneous or confounding variables. A multiple case study might eliminate some validity threats of the findings, but the variation of key variables across multiple cases needs to be explicitly addressed in interpreting the findings. In addition, the selection of multiple cases needs to be matched with the problem under investigation while controlling other variables. This dissertation study views that one of the greatest predicaments of teaching is its dependence on students. By holding other variables such as teacher and content relatively constant while only varying the students, this dissertation is designed to untangle the ways in which the same teacher adjusts instruction to meet the students' needs in developing mathematical explanation.

Lastly, the conceptual framework developed in this dissertation, which decomposes the work of teaching into four core tasks of teaching, anchors in the key components of the instructional triangle. The four core tasks of teaching are not just a random collection of observations conveniently or randomly extracted from the data, but are deliberately devised to structurally capture the crucial elements of instructional interactions. Taking serious account of the instructional triangle, the four core tasks of teaching anchor in both mathematics and students, but vary with the degree of complexity in regarding to the demand of unpacking students' ideas in a public space. The next section introduces the organization of this dissertation study.

1.8. Organization of the Dissertation

This dissertation is organized into ten chapters. The current chapter, Chapter 1, introduces the problem statement that motivates this dissertation study and then provides an overview of this dissertation study. To articulate components of what is entailed in the work of teaching to support students to develop mathematical explanation, Chapter 2 discusses four key constructs of this dissertation: (1) the work of teaching; (2) mathematical explanation; (3) instructional resources; and (4) mathematical task. Chapter 3 describes the study design, data sources, data analysis, and limitations of this study.

Chapters 4 through 7 present an extensive detailed analysis of instructional interactions managed by the same teacher for teaching the same mathematical tasks to different cohorts of students, particularly focusing on supporting students to develop mathematical explanation. Chapter 4 provides a detailed analysis of the brown rectangle problem across five years and Chapter 5 provides a detailed analysis of the blue and green rectangle problem across five years. Chapter 6 provides a detailed analysis of the two-coin problem across two years and Chapter 7 provides a detailed analysis of the three-permutation problem across three years. The themes that emerged from these four analysis chapters are somewhat similar, but are conceptually distinguishable at the same time; that is, Chapters 4 and 5 uncover the features of developing mathematical explanation for definition whereas Chapters 6 and 7 uncover the features of developing mathematical explanation for the structure of multiple solutions.

Building on these individual cases analyses, the first part of Chapter 8, as a cross-year analysis, discusses the relationship between instructional features (instructional contexts, pedagogical approaches, mathematical approaches, and students' ideas) and constructing mathematical explanation. The second part of Chapter 8, as a cross-mathematical-task analysis, argues the distinction between "mathematical-task generality" and "mathematical-task specificity" of constructing mathematical explanation. Building on these analytical grounds, Chapter 9 introduces a conceptual framework for the work of teaching entailed in supporting students to develop mathematical explanation around four core tasks of teaching and two instructional resources to support that work. Lastly, Chapter 10 provides a summary of this dissertation study, demarcates this dissertation, discusses the contribution of this dissertation study in terms of the research design for studying teaching and the method of decomposition of teaching, and then concludes with the implications of this dissertation study for teacher education.

CHAPTER 2.
THEORETICAL FOUNDATIONS

2.1. Overview

This dissertation aims to conceptualize *the work of teaching* entailed in supporting students to develop *mathematical explanation*, and particularly, the ways of using the two key *instructional resources* of *discourse resources* and *collective resources* to that end. The work of teaching refers to core tasks to be undertaken in the process of managing multiple relationships simultaneously in order to produce student learning (Ball & Forzani, 2009; Cohen, 2011; Lampert, 2001; Thames, 2009). Following an approach of decomposing the complexity of teaching into its constituent components (Grossman et al., 2009), this dissertation articulates components of what is entailed in the work of teaching to support students to develop mathematical explanation. This chapter discusses four key constructs of this dissertation: (1) the work of teaching; (2) mathematical explanation; (3) instructional resources; and (4) mathematical task.

2.2. A Review on the Work of Teaching

2.2.1. A Need for Developing a Theory of Teaching

A theory plays an important role in that it provides a lens to explore the phenomenon, aids in the formulation of research questions, contributes to explanations of the relationships being studied, and builds the logical structures of findings (Creswell, 2013; Hiebert & Grouws, 2007; Kerlinger, 1979; Parsons, 1938). Despite its importance, a theory is underdeveloped in research on teaching. Hiebert and Grouws (2007) point out that theories of learning are relatively well articulated, but they do not directly transfer to theories of teaching. Even though theories of learning provide foundations for understanding how students learn and for organizing effective learning environment for students, they are insufficient for teachers to manage the complexity of instructional interactions. Hiebert and Grouws (2007) argue that the recent efforts to identify key features of teaching show significant progress toward developing theories of teaching, but theories of teaching need to be further developed ways "in which the key components of teaching fit together to form an interactive, dynamic system for achieving particular learning goals." (p. 373).

Over the decades, researchers have characterized teaching in a number of different ways depending on their theoretical orientations, substantive issues, and practical purposes: complex cognitive skill (Leinhardt & Greeno, 1986), improvisational performance (Borko & Livingstone, 1989), and decision making (Hunter, 1979; Shavelon, 1973; Schoenfeld, 2011), to name a few. Acknowledging a variety of views, the following sections aim to characterize teaching that provides a foundation for this dissertation study. In doing this, my purpose is to explicate a lens to study teaching. This section begins with a brief overview of terms that are often adopted to explain teaching and then characterizes what I mean by teaching. In order to describe and analyze the work of teaching, the following section then identifies the key components of teaching and describes the complicated relationship among them.

2.2.2. Four Constructs of Teaching

It is important to clarify what is meant by teaching because the familiar term "teaching" can be interpreted in a wide variety of ways according to one's knowledge, beliefs, and experience. This section begins by examining the dictionary definitions of teaching because "a dictionary definition provides the minimal information necessary to evoke the concept it defines in the mind of a human reader who already knows to what this concept refers." (Amsler, 1998, p.458) The Oxford Dictionary defines teaching as "the occupation, profession, or work of a teacher," the American Heritage Dictionary defines teaching as "the act, practice, occupation, or profession of a teacher, " and the Cambridge Dictionary defines teaching as "the job of being a teacher." These dictionary definitions suggest four constructs related to teaching that are closely related but often undifferentiated in the field: (1) the occupation of teaching; (2) the profession of teaching; (3) the practice of teaching; and (4) the work of teaching. By examining the lexical meanings and disciplinary usages of these constructs, this section aims to clarify what I mean by teaching and then to elicit key features of teaching; otherwise the familiar term—teaching—could be interpreted in a wide variety of ways according to one's knowledge, beliefs, and experience.

First, teaching is often considered to be what people in the occupation of teaching normally do, say, or think. This approach is well articulated by Lortie (1975) in his seminal sociological study of teaching. From the occupational perspective, Lortie searches for the pattern of orientations and sentiments that are prevalent among teachers by examining the processes of recruitment, socialization, and career rewards that perpetuate the structure of occupation and analyzes teachers' personal goals, uncertainties, and interpersonal preferences in which they bring meaning to the occupation. Using these structural and phenomenological approaches, Lortie characterizes teaching as an occupation that orients conservatism, individualism, and presentism, but argues that it needs to be changed in a way that accommodates adaptability, collegial responsibility, and technical knowledge. One of the aspects that many researchers have shed light on is building the collective knowledge of the occupation in order to improve the individualized perception of teaching. For instance, as a way of generating, accumulating, and sharing knowledge about teaching, Hiebert, Gallimore, and Stigler (2002) illustrate how Japanese lesson study transforms teachers' individual craft knowledge into collective professional knowledge.

The second construct adopted to explain teaching is a profession. The American Heritage Dictionary defines profession as (1) an occupation or career; (2) an occupation, such as law, medicine, or engineering, that requires considerable training and specialized study; and (3) the body of qualified persons in an occupation or field. The first definition uses an occupation as a synonym for a profession, the second definition refers to the subset of occupations which require specialized knowledge and skills, and the third definition refers to the subset of persons in the occupation who have some qualifications. Either of those meanings—the subset of occupations or the subset of people in the occupation—suggests that teaching requires specialized knowledge and skills. On the one hand, researchers have theoretically conceptualized knowledge needed for teaching (Shulman, 1986; Ball, Thames, & Phelps, 2008) and have provided empirical evidence of the kinds of knowledge that matter for instructional quality (Hill, Ball, & Schilling, 2008) and student achievement gains (Baumert et al., 2010; Hill, Rowan, & Ball, 2005). Other researchers have investigated the features of expert teachers. For instance, Leinhardt and Steele (2005) analyze the complexity of instructional dialogue by an expert teacher and describe how the expert teacher makes use of tools—routines, meta-talk, and intellectual climate—to support the instructional dialogue.

The next construct used to account for teaching is practice. The Oxford Dictionary defines practice as (1) the actual application or use of an idea, belief, or method as opposed to theories about such application or use; and (2) repeated exercise in or performance of an activity or skill so as to acquire or maintain proficiency in it. Beyond these dictionary definitions in general, Lampert (2010) judiciously clarifies four different meanings of practice that have been used in the literature: (1) as a contrast to theory, (2) a collection of practices which could be further decomposed, (3) rehearsal for attending proficient performance, and (4) shared collective identity in community. She then discusses how each concept has been used for designing teacher education programs.

In studying teaching and its predicaments, Cohen (2011) discusses that the occupation of teaching has been examined from various angles, but little is known about the work of teaching. His comment draws attention to the meaning of work. The Oxford Dictionary defines work as (1) activity involving mental or physical effort done in order to achieve a purpose or result: (2) mental or physical activity as a means of earning income; employment; (3) a task or tasks to be undertaken; something a person or thing has to do; and (4) something done or made. Instead of the meaning that indicates activities done or undertaken to make a living, this dissertation study focuses on the phrase "achieve a purpose or result" in order to highlight that the work entails the goal-driven activities.

Although many researchers have interchangeably used the work of teaching with the practice of teaching, a detailed examination of these two terms reveals subtle differences. These lexical meanings signify that the term *practice* refers to habitual, customary, and repeated actions to acquire proficiency in teaching but the term *work* refers to tasks involving conscious, concerted, purposeful, and intentional efforts to produce, achieve, or accomplish something. In this sense, this dissertation study uses the term "the *work* of teaching" because it specifies the goals, efforts, and instructional components. Several scholars write:

> We claim that *practice* must be at the core of teachers' preparation and that this entails close and detailed attention to the *work* of teaching and the development of ways to train people to do that *work* effectively, with direct attention to fostering equitably the educational opportunities for which schools are responsible. By "*work* of teaching," we mean the core tasks that teachers must execute to help pupils learn. (Ball & Forzani, 2009, p. 497, italics added)

> I will explore different meanings of *practice* and the different implications each has for how one learns or gets better at the *work* of teaching. (…) [w]e are particularly concerned with teaching that occurs in school classrooms, where the *work* entails responsibility for whole classes of students compelled to work together for 9 months at a time. (Lampert, 2010, p.22, italics added)

These statements indicate that the term *work* implies taking responsibility for achieving its goal, whereas the term *practice* provides implications for preparing, training, and learning the *work*. In this sense, this dissertation uses the term "the *work* of teaching" rather than "the *practice* of teaching" because this dissertation mainly focuses on the nature of teaching rather than on learning to teach[8]. Certainly, the results of this dissertation study contribute to the design of teacher education programs and professional development programs to learn the work.

Because of its purposeful and intentional goal-driven activity, the work of teaching can be distinguishable from everyday informal activities of knowledge-in-transit. Thus, it is unnatural (Ball & Forzani, 2009), deliberate (Thames, 2009), and attentive (Cohen, 2011). The next section examines how teaching is conceptualized in the instructional context and reviews challenges for teachers to do the work.

2.2.3. Teaching as Managing Multiple Relationships in a Responsible Way

In outlining a theory of instruction, Cohen, Raudenbush, and Ball (2003) define instruction as "interactions among teachers and students around the content, in environments" (p.122) denoted by bidirectional arrows in Figure 2.1. Later, highlighting the role of teachers in managing such interactions, Cohen (2011) explains that "teaching consists not in what teachers know, but in what they know how to do with students and what students know how to do themselves, with one another, with some content, and with their teachers in their environment." (p.51)

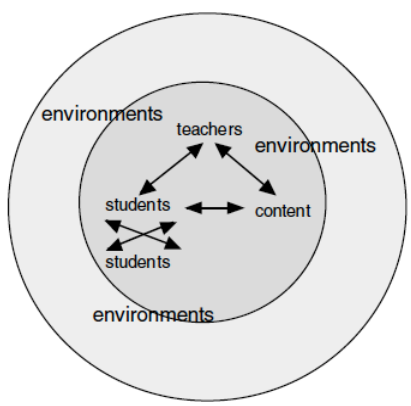


Figure 2.1. Instruction as Interaction (Cohen et al., 2003, p.124)

As a precursor of this simplified version of instructional dynamics, Lampert (2001) represents more nuanced, detailed, and complicated versions of instructional

[8] Lampert (2010) uses the term "learning teaching" rather than "learning to teach" because the former suggests that learning occurs while doing the work, whereas the latter implies that the action is occurred in the future after the work is learned.

dynamic. Lampert first identifies three problem spaces—a bidirectional arrow between a teacher and a student, a bidirectional arrow between a teacher and content, and a bidirectional arrow between a teacher and student-content relationship—and then merges these three arrows into a single three-pronged space (Figure 2.2).

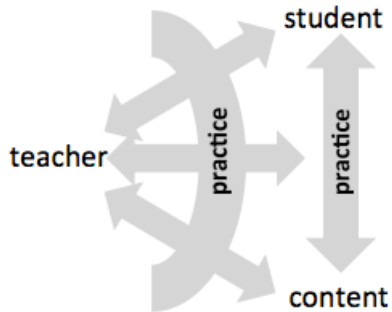


Figure 2.2. A basic three-pronged representation of teaching practice (Lampert, 2001, p.33)

By doing this, Lampert represents the idea that each relationship is not separate but interconnected. She explains that "the work of teaching is done in simultaneous relationships with students, with content, and with the student-content connection, while students do the complementary work of making relationships with the content to learn it." (p.423) Lampert further elaborates the basic three-pronged representation of teaching by considering temporal, social, and content complexities. As teaching involves interactions with more than one student in the classroom, the basic three-pronged representation is extended to include not only a practice-arrow between a teacher and a student, but also practice-arrows between a teacher and groups of students and between a teacher and a class as a whole, in relationships with the content as the whole terrain of a conceptual field rather than a single topic (Figure 2.3).

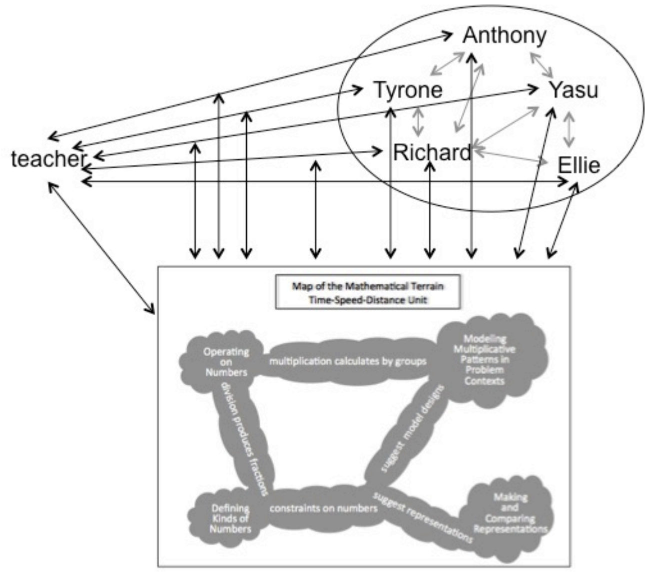


Figure 2.3. An elaborated representation of teaching practice (Lampert, 2001, p.437)

In managing these multiple relationships simultaneously, one relationship might conflict with another relationship (Ball, 1993; Lampert, 2001). On one hand, supporting students' participation, eliciting their multiple solutions, or honoring their thinking can decrease opportunities to focus on the important mathematical ideas or leave incorrect mathematical ideas unexamined without further investigations. On the other hand, over-attending to the mathematical ideas can close the door to hearing students' thinking, jeopardize their participation, and undermine their authority. In addressing this concern, Lampert writes:

> Each relationship both limits and expands what the teacher can do to address problems of practice. Interactions with students get in the way of relating to the content, and the content gets in the way of relationships with students. While the teacher is trying something to get students interested in content, she may be doing things that interfere with her own understanding of the content. Such conflicts must be managed in each act of teaching. (Lampert, 2001, pp. 33-34)

Recognizing the challenges that teachers face with these contradictory goals, researchers have examined ways in which teachers resolve this tension. For instance, Sherin (2002) illustrates how one middle school teacher, Daniel, continuously struggled with balancing between students and content over the year but balanced them at times by using "filtering approach"—first eliciting students' multiple ideas, comparing ideas, and filtering ideas. The efforts to create a mathematical discourse community have been focused on balancing these two components of instruction wherein one component limits the other, but it has not sufficiently examined yet how to coordinate these two components harmoniously in a way in which one component expands the other. The next section addresses an approach of decomposition in research on teaching and teacher education.

2.2.4. Decomposing the Work of Teaching

Teaching is often described as a complex activity that involves managing multiple relationships simultaneously with students and with content over time. To make this complex work doable and learnable for teachers, especially for beginning teachers, several scholars (Boerst, Sleep, Ball, & Bass, 2011; Grossman et al., 2009; Sleep, 2009) have addressed the need to decompose the work of teaching into its constituent components. As a result, some domains of the work of teaching have begun to be decomposed into nested practices with varying grain sizes (e.g., Boerst et al., 2011; Sleep, 2009).

Despite these initiatives, the call for "a specific technical language for describing the implicit grammar and for naming the parts" (Grossman et al., 2009, p. 2069) has not met an agreed-upon robust framework yet. For instance, in decomposing the work of steering instruction toward the mathematical point, Sleep (2009) first identified seven central tasks: (1) attending to and managing multiple purposes; (2) spending instructional time on mathematical work; (3) spending instructional time on the intended mathematics; (4) making sure students are doing the mathematical work; (5) developing and maintaining a mathematical storyline; (6) opening up and emphasizing key mathematical ideas; and (7) keeping a focus on meaning. Each core task is further decomposed into strategies and problematic issues, wherein some strategies (e.g., strategically selecting numbers for examples and exercises) are the opposite statement of problematic issues (e.g., nonstrategic selection of numbers in examples or exercises). These central tasks are neither sequential nor mutually exclusive; rather, they might be enacted simultaneously. For each core task, Sleep further decomposed it into strategies and problematic issues, but did not associate the core tasks with particular teacher moves. Drawing from theories of distributed cognition, Sleep (2009) clarifies that the work of designing instruction is "likely to be differently distributed, and models of the distribution of work could be made for these particular situations" (p.23).

On the other hand, in decomposing the work of leading a mathematical discussion, Boerst et al. (2011) started with the larger grain size of domains (e.g., leading a discussion) and then specified it into the smaller grain size of techniques (e.g., revoicing), while articulating intermediate practices (e.g., clarifying student thinking) that connect between domains and techniques. Unlike the method that Sleep (2009) used, Boerst et al. (2011) decomposed the work of leading a discussion into five stages in a chronological order: (1) setting up the problem; (2) monitoring student work; (3) launching the discussion (4) orchestrating the discussion; and (5) concluding the discussion. They made a direct connection between techniques and practices, by examining how a particular practice could be accomplished through different techniques and why a particular technique is used to serve the purpose. For instance, Boerst et al. (2011) explain that the technique of revoicing serves the purposes of following up student responses, clarifying student thinking, facilitating connections, and encouraging attentions to the contribution of others, while the intermediate practice of clarifying student thinking could be accomplished by different techniques such as revoicing, requesting a representation, asking a probing question, and having the student talk about the related problem.

In summary, the issues around the structure of decomposition (e.g., a sequential structure or a layered structure), the level of decomposition, and the link to the teacher moves or discourse moves are not well articulated and discussed yet in research on teaching and teacher education. In addition, naming the constituent components is more generic, but does not reflect the nature of the discipline. The next section reviews the related concepts to mathematical explanation used in the literature and provides the specific criteria for a disciplinary-grounded mathematical explanation in the context of classroom community conceptualized in this dissertation study.

2.3. A Review on Mathematical Explanation

This section reviews three related lines of research on mathematical explanation: (1) situated in a spectrum of mathematical practices; (2) identified as a special genre of response to a query; and (3) mathematical explanation vs. discussion. After situating the term "mathematical explanation" in these different contexts, this section provides the specific criteria of mathematical explanation conceptualized in this dissertation.

2.3.1. Mathematical Explanation Situated in a Spectrum of Mathematical Practices: Mathematical Explanation, Justification, Argumentation, and Proof

Mathematical explanation, justification, argumentation, and proof have drawn attention as core mathematical practices to be cultivated in mathematics classrooms. Despite the increased interest in these mathematical practices across a large body of literature, a clear distinction has not been well made. Often, it is viewed that explanation demands less formality, rigor, logical structure, and deduction, thus the activity of explaining is more appropriate for lower-level elementary students, whereas proof demands more formality, rigor, logical structure, and deduction thus the activity of proving is more appropriate for high school or college students. Whereas a number of studies have developed conceptual framework for justification (Simon & Blume, 1996; Staples, Bartlo, & Thanheiser, 2010), argumentation (Cobb, Wood, Yackel, & McNeal, 1992; Forman & Larreamendy-Joerns, 1998), and proof (Knuth, 2002a; Knuth, 2002b; Stylinanides, 2007) for teaching and learning mathematics in classrooms, mathematical explanation has often been equated with discursive activities. This section briefly reviews how these related mathematical practices are conceptualized in the literature.

Yackel and Cobb (1996) illustrate how a teacher and students interactively establish what counts as an acceptable explanation in an inquiry mathematics classroom. They illustrate three aspects of what counts as an acceptable mathematical explanation: (1) mathematical basis for explanations rather than social basis; (2) explanations as descriptions of actions on experientially real mathematical objects rather than following procedural instructions; and (3) explanations as objects of reflection. They view that explanation serves as "clarifying aspects of one's (mathematical) thinking that might not be apparent to others" whereas justification is provided "in response to challenges to apparent violations of normative mathematical activity." Compared to explanation and justification that serve as communicative functions, Yackel and Cobb (1996) view argumentation as "a logical structure including conclusion, data, warrants, and backing" using Tolumn's (2003) argumentation tool. In a similar vein, using Toulmin's approach, Forman and Larreamendy-Joerns (1998) defines argumentation as "the intentional explication of the reasoning of a solution including claims, grounds, warrants, and backings" and Wood (1999) define argumentation as "a discursive exchange among participants for the purpose of convincing others through the use of certain modes of thought" (p.172)

Another line of research has focused on the relationship between explanation and proof in the discipline. Several scholars have argued whether proof counts as explanation or not. Hersh (1993) argued that every proof counts as explanation, whereas Mancosu (2001) argued that the full proof might not be considered as explanations in the classrooms. Mancosu (2001) argued that a picture or informal arguments provide explanations, but proof does not. Making a distinction between explanation for instructions and explanation for an account of reason why, Hafner and Mancosu (2005) summarize some characterizations of explanation used by mathematicians and philosophers including deep reasons, an understanding of the essence, a better understanding, a satisfying reason, the reason why, the true reason, an account of the fact, and the causes of. On the other hand, Balacheff (1988) defines proof as an explanation which is accepted by a community at a given time, whereas the term explanation is used to describe the discourse of an individual to establish the validity of a statement for somebody else. He conceptualized mathematical explanation in a way of constructing meaning individually, whereas mathematical proof as a nature of accepted knowledge in the community.

A few studies have made an effort to make a distinction, especially between argumentation and proof (Balacheff, 1999; Boero, 1999; Mariotti, 2000; Knuth, 2002b; Pedemonte, 2007). Knuth (2002b) defines proof as "deductive argument that shows why a statement is true by utilizing other mathematical results and/or insight into the mathematical structure involved in a statement," whereas argument as "referring to nonproof.” (p.86) Mariotti (2000) refers proving to “both logically enchained arguments which are referred to a particular theory” whereas refers argumentation to “remove doubts about the truth of an statement” (pp.30-31). Pedemonte (2007) argues that the logical structure of proof is deductive, whereas the logical structure of argumentation is unlikely to be deductive steps, such as abductive steps or inductive steps (p.29). Despite the claim that proof has a different nature from argument, these terms are often used interchangeably without a clear distinction in many other contexts.

Stylianides (2007) argues that proof is a deductive argumentation and excludes empirical argumentation with the following three characteristics: (1) set of accepted statements; (2) modes of argumentation; and (3) mode of argument representation. Using Stylianides’ conceptualization, Staples (2012) defines justification as “an argument that demonstrates (or refutes) that uses accepted statements and mathematical forms of reasoning” (p.448) because it is more appropriate to discuss mathematical work by middle grade students. On the other hand, Simon and Blume (1996) define justification as “a cognitive and a social process, the process of working within socially constituted and accepted modes of establishing validity to collectively determine what is cognitively compelling” (p.28). Table 2.1 summarizes the use of explanation, justification, argumentation, and proof in the selected literature.

Table 2.1. The use of explanation, justification, argumentation, and proof in the selected literature

	Explanation	Justification	Argumentation	Proof
Yackel & Cobb (1996)	×	×	×	
Forman et al. (1998)			×	
Wood (1999)			×	
Hersh (1993)	×			×
Mancosu (2001)	×			×
Hafner & Mancosu (2005)	×			
Balacheff (1988)	×			×
Pedemonte (2007)			×	×
Mariotti (2000)			×	×
Knuth (2002b)			×	×
Stylianides (2007)				×
Staples (2012)		×		
Simon & Blume (1996)		×		

In a brief review of the literature, the nature of mathematical explanation is not well distinguished from other mathematical practices such as mathematical justification, argumentation, and proof. The value of mathematical explanation is often underestimated because it is considered as an entry level of mathematical practices in the spectrum of mathematical practices. In addition, the features of mathematical explanation are not well differentiated from other verbal, linguistic, discursive, and communicative acts.

2.3.2. Mathematical Explanation Identified as a Special Genre of Response to a Query: Common Explanation, Self Explanation, Disciplinary Explanation, and Instructional Explanation

Mathematical explanation has been also identified as a special genre of response to a query, while contrasted with other types of explanations in different contexts. Leinhardt (2001) makes a distinction between common explanation, self-explanation, disciplinary explanation, and instructional explanation. She argues that the four types of explanation both share certain features (e.g., depending on an explicit or implicit query; conforming to specific rules of closure or completeness; having certain regularities with respect to evidence and audience) but they also differ in the specifics of these features (e.g., the specific type of query; the specific kind of evidence; the specific sense of audience; and the specific rules for closure).

Leinhardt (2001) explains that the appropriateness of common explanation is determined by the degree of satisfaction by an inquirer and social system of discourse but it does not require specialized language or form of reasoning; disciplinary explanation adheres to precise conventions of completeness and closure, have a more tacit set of conventions about what constitutes a legitimate question, what the agenda for the discipline is, what is required for evidence to be accepted, and what the rules for refutation are; self-explanation is colloquial, personally referential, fragmentary, and idiosyncratic; and instructional explanation is more exhaustive than the verbal traces of self-explanation but it is less formal and more redundant than disciplinary explanation. She further explains that common explanation is the default condition of instructional explanation and disciplinary explanation set as upper bound of instructional explanation, thus instructional explanation plays a role of bridging the gap between the common explanation and disciplinary explanation. One major difference between instructional explanation conceptualized by Leinhardt (2001) and supporting students to develop mathematical explanation conceptualized in this dissertation is that Leinhardt (2001) views that instructional explanation can be given by a textbook, a teacher, a student, or be jointly built through discourse whereas this dissertation focuses more on the collective construction of mathematical explanation by a group of students with a teacher's support.

2.3.3. Developing a Mathematical Explanation vs. Leading a Whole-Group Discussion

The work of teaching entailed in supporting students to develop mathematical explanation is interrelated with leading a whole-group discussion—both conceptually and practically—, thus the boundary between them might not be so clear. The reasons for choosing the term "supporting students to develop mathematical explanation" rather than "leading a whole-group discussion" are as follows.

First, a consensus has not been reached yet on what constitutes a discussion, how to distinguish a discussion from other instructional discourse formats, and what criteria might be applied to evaluate the quality of discussion. Leading a whole-group discussion has been characterized by the quantity of student talk, the participation structure, the degree of authorities and responsibilities shared to students, or the use of certain verbal indicators such as "agree," "disagree," and "argue." On the other hand, when a teacher plays a significant role in shaping the structure of discussion, it is not often viewed as a leading a discussion.

Second, it is not often clear what kind of discussion is expected for a given mathematical task. The mathematical elements are often implicit or unanticipated in the term "leading a whole-group discussion," while leaving a space for any topics that are discussable for the given mathematical task. For instance, in case of the two-coin problem, an extensive verbal exchange might occur about the probability of pulling out a penny, nickel, or dime, instead of developing mathematical explanation around proving whether each proposed solution meets the conditions of the problem or whether all solutions are found. This might be a productive discussion or a mathematically valuable exchange, but the discussion around the probability issue derails the development of the key mathematical ideas of the given mathematical task. In decomposing the work of leading a whole-group discussion, Boerst et al. (2011) address the similar concern in characterizing a discussion. They write:

> First, for a mathematics discussion to provide the opportunity for participants to engage in sustained reasoning, it must have a focus. This focus may be a concept, procedure, problem, or recently completed activity that can evolve through interaction. This focus has sufficient depth to support dialogue for an extended period. (…) Third, the teacher is an active participant in a discussion and not a passive observer. The teacher contributes ideas and encourages students to participate, attend to the contribution of others, and engage with the mathematical focus of the discussion. (p.2847)

Because the term discussion could miss the key mathematical points of the given mathematical task or be abounded with non-mathematical ideas of the given mathematical task, this dissertation uses the term "supporting students to develop mathematical explanation" in order to more directly address the core set of mathematical ideas that needs to be explained and to achieve the instructional goals for the given mathematical task. Building on the issues raised in the review of three related lines of research on mathematical explanation, the following section provides the specific criteria of mathematical explanation conceptualized in this dissertation.

2.3.4. Conceptualization of Mathematical Explanation

Given that the practice of explanation has often been equated with general discursive activities, just increasing the amount of talk might be less likely to support the construction of knowledge required by mathematical work[9]. Instead of using the term "explanation" as more general discursive activities, this dissertation deliberately uses the term "mathematical explanation" to describe the specifications of mathematical reasoning for answering why-questions entailed in the mathematical task. This section first reviews the features of good mathematical explanations that emerged from an analysis of making and evaluating mathematical explanation in the context of teacher education class and

[9] This argument is in a line with Balacheff's (1999) argument between argumentation and proof. Balacheff argued that argumentation does support the modification of discourse structure, but it does not support the modification of the status and the functioning of the knowledge required by mathematical work.

then introduces four criteria adopted to conceptualize mathematical explanation in this dissertation study.

Instead of epistemologically or philosophically defining the concept of mathematical explanation in relation to other related concepts, in the context of teacher education class, Ball (2004) takes a more pragmatic, practical, and practice-based approach to discuss the features of good mathematical explanations with preservice elementary school teachers in her elementary mathematics content course. By analyzing how students make and evaluate a mathematical explanation for a particular mathematical task as a form of justification about why a mathematical claim is true, the class produced a list of features of good mathematical explanations for the "cooking jar problem" (see Figure 2.5).

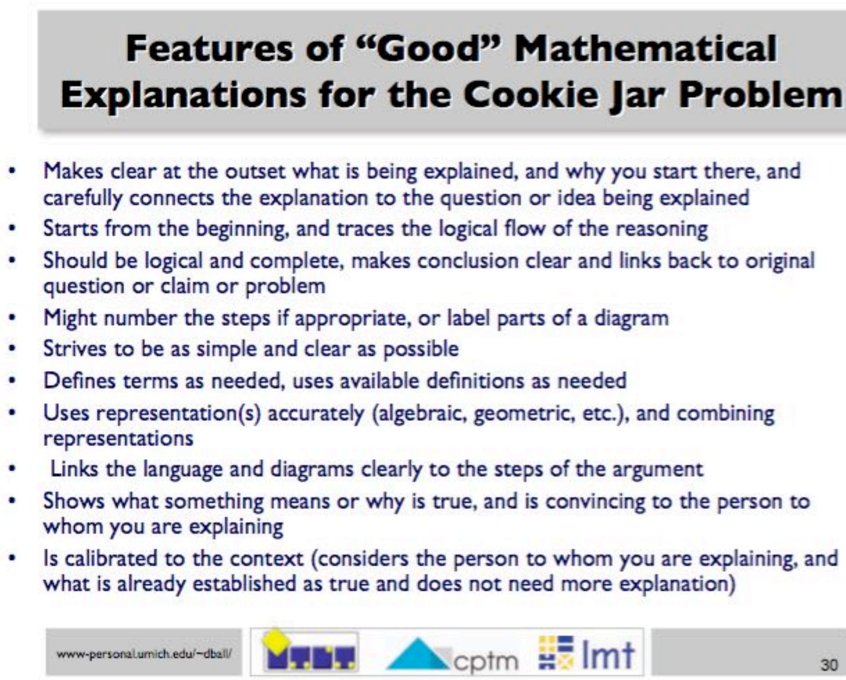


Figure 2.4. Features of "good" mathematical explanations for the cookie jar problem (Ball, 2004)

Taking a similar approach, this dissertation conceptualizes the features of mathematical explanation through the iterative analysis of making, evaluating, and responding to explanation made by students in the classroom setting. As a result, I conceptualize a more disciplinarily grounded mathematical explanation in the context of a classroom community with the following four specific criteria: (1) explicating the key ideas, conditions, constraints, or structure embedded in mathematical task; (2) articulating an explanation in a way that reduces ambiguity, incoherence, inconsistency, inaccuracy, and unnecessary redundancy by utilizing representations, examples, or counterexamples; (3) grounding the legitimate, valid, and logical form of knowledge in the discipline; and (4) grounding the publicly accessible, available, and acceptable knowledge that is shared in the classroom community.

The last two criteria of this conceptualization resonate with the characteristics of proof proposed by Stylianides (2007), but my four criteria can be differentiated for the following two reasons. First, the mathematical explanation conceptualized in this dissertation is more specific to the nature of mathematical task rather than generic features such as the number of answers (e.g., only one answer vs. multiple answers) or the problem statement (e.g., verifying vs. refuting). Second, this conceptualization somewhat alleviates the requirement of refutation. Indubitably, the refutations made by other students enrich and expand the warrants of mathematical explanation but they are not an indispensible aspect of mathematical explanation. One caution is that this dissertation does not aim to evaluate whether each statement provided by an individual student at a particular instructional moment can be counted or acceptable as mathematical explanation or not (c.f., Yackel & Cobb, 1996). Rather, this dissertation focuses on how to support students to develop a mathematical explanation that meet the above criteria.

2.4. A Review on Instructional Resources

In doing the work of teaching, teachers can utilize instructional resources. Shifting from the research programs that considered conventional resources (e.g., class size) as important for student learning, it has been argued that the effect of resources depends on how they are socialized in the interactional dynamic of instruction (Cohen et al., 2003; Cohen, 2011). There are many resources that could be utilized in the work of teaching, but this dissertation study examines the two key instructional resources that are mobilized through the instructional interactions: discourse resources and collective resources.

2.4.1. The First Key Instructional Resource: Discourse Resources

With the increased interests in creating mathematics classroom discourse, researchers have proposed discourse moves as crucial pedagogical strategies. For instance, Chapin and O'Connor (2004) propose revoicing, asking students to restate someone else's reasoning, asking students to apply their own reasoning to someone else's reasoning, prompting students further participating, and using waiting time; Ghousseini (2008) proposes revoicing, orienting students to each other, pressing students for explanation, connecting students' ideas and negotiation, and modeling or pointing to specific aspects of the discourse; Herbel-Eisenmann, Drake, and Cirillo (2009) propose revoicing, asking students to revoice, inviting student participation, probing a student's thinking, and creating opportunities to engage with another's reasoning, and waiting. In general, these discourse moves have increased students' discursive activities in mathematics classrooms.

Based on my personal experiences of observing mathematics lessons in a variety of settings, however, it is observed that teachers often injudiciously, overly, or inappropriately use these discourse moves without knowing how they specifically function to enrich the construction of knowledge in a disciplinary meaningful way. Even though the arguments about the effect of discourse moves are made in the context of teaching and learning mathematics, they are often free from the issues that only matter for the specific subject-matter, not highlighting the distinctive features of the discipline of mathematics. Still, the question still remains about how a particular discourse move (e.g., revoicing) is used to address the specific mathematical issue and how different mathematical issues require the employment of different discourse resources. In the analysis of classroom discourse using systemic functional linguistics, Herbel-Eisenmann and Otten (2011) express concerns about the "content-free" analysis of mathematics classroom discourse. Taking the approach that "context and language are intimately related: Context influences language choice and language choice helps to construct context" (p. 452), they address the need to focus on how mathematics is construed in the classroom discourse.

This dissertation study does not utilize systematic functional analysis, but parallels in the idea that mathematical issues influence language choice. The previous studies particularly pay attention to how authority, identity, and responsibility influence language choice, but this dissertation study more directly pays attention to how mathematical issues function in the choice of discourse resources. Instead of suggesting particular discourse moves, this dissertation investigates the choice of discourse based on mathematical issues.

2.4.2. The Second Key Instructional Resource: Collective Resources

In addition to discourse resources, collective resources are another key instructional resources served for the work of teaching because instruction is not just targeted for one single individual student but it is processed through dynamic interactions between students in the classroom. The practice of mathematical explanation in the discipline is produced for the community, not for any particular individual or authority figure. Likewise, the practice of mathematical explanation in a classroom is intrinsic to the classroom community, not just for the teacher to hear, comment, and critique. In this sense, in cultivating the practice of mathematical explanation in classroom, the collective work plays an important role as a central instructional resource. There is a large body of literature on group work in mathematics classrooms, but this dissertation focuses on the collective work, in contrast to the notions of collaborative and joint activities that are commonly used in the literature. After examining the effect of collaborative learning and the process of collaborative learning, this section elaborates on the meaning of collective work.

In one line of work, using terms such as "collaborative learning" or "cooperative learning", researchers have investigated the conditions that are effective for small-group interactions (Johnson & Johnson, 1975; Watson& Chick, 2001; Webb, 1991). For instance, in a meta-analysis of 17 empirical studies on small-group interactions, Webb (1991) showed that some compositions of small-groups (e.g., homogenous group with medium-ability students; heterogeneous group with medium-ability and low-ability students; heterogeneous group with high-ability and medium-ability; and groups with equal numbers of boys and girls) have greater positive effects on students' mathematics achievement than other compositions (e.g., homogenous group with high-ability students; homogenous group with low-ability students; heterogeneous group with high-ability;

medium-ability and low-ability students; and groups with unequal numbers of boys and girls). Watson and Chick (2001) examined cognitive, social, and external factors associated with the effectiveness of collaborations on solving open-ended problems. Even though these studies mainly focused on the interaction between students in the context of small-group activities with the minimal intervention of a teacher during instruction, they showed some potential benefits of encouraging verbalization, increasing responsibility, and solving problems in various ways.

In another line of work, researchers have examined the construction of meaning through joint activities. Instead of extracting effective conditions of cooperative learning that lead to the improvement of students' mathematical achievement, Yackel, Cobb, and Wood (1991) focused on the process of cooperative learning. Labeling as "second-generation" of cooperative learning by themselves, they focused on the mutual construction of classroom norms for cooperative learning. They further elaborated three types of learning opportunities occurring in small-group interactions: (1) opportunities to use other students' solutions as prompts in developing one's own solutions; (2) opportunities to reconceptualize a problem for the purpose of analyzing an error; and (3) opportunities to extend one's conceptual framework to make sense of another's solution for reaching consensus.

This dissertation examines the collective development of publicly acceptable knowledge in the classroom community. In this sense, this dissertation uses the term "collective work" which is conceptualized by Ball and Bass (2003) rather than other terms, such as "collaborative learning" or "joint activities," because the former highlights the process of knowledge construction in the community including generating, listening, critiquing, justifying, validating, and certifying based on the disciplinary reasoning. As for the benefits of using collective work, Ball (1993) writes:

> [I] aim to develop each individual child's mathematical power through the use of the group. I aim to develop the children's appreciation for and engagement with others different from themselves. … The classroom community is often, as the children themselves note, a source of mathematical insights and knowledge. The students hear one another's ideas and have opportunities to articulate and refine or revise their own. Their confidence in themselves as mathematical knowers is often enhanced through this discourse. (p.394)

In a similar vein, Cohen (2011) argues that individual student knowledge could be a social resource for others, depending on how this knowledge is made accessible to others. As teaching involves dynamic interactions with groups of students, not just with a single individual student, collective work is often described as a challenge in the work of teaching. However, the resource that each individual student brings to the class can serve as another resource for other students to elaborate, articulate, revise, expand, and validate their mathematical reasoning.

2.5. A Review on Mathematical Tasks

Mathematical tasks provide opportunities for students to learn mathematics. In setting a vision of teaching mathematics, the *Professional Standards for Teaching Mathematics* (NCTM, 1991) views mathematical tasks as a means to "convey messages about what mathematics is and what doing mathematics entails" (p. 24) and argues that teachers need to select, adapt, or develop worthwhile mathematical tasks while considering mathematical content, students, and the ways in which students learn mathematics.

Beyond the importance of having worthwhile mathematical tasks, several scholars have examined how mathematical tasks are implemented in instructional interactions. Using the term academic task, conceptualized as a classroom process rather than a context variable, Doyle and his colleagues (Doyle & Carter, 1984; Doyle, 1986a; Doyle, 1986b; Doyle, 1988) argue that the tasks assigned by a teacher determine how students understand the curriculum. By categorizing the academic tasks into familiar tasks and novel tasks, they analyzed how each type of task unfolded in instruction. More specifically, when familiar work (low cognitive demand) was done, the flow of classroom activities was smooth, the involvement of students was high, and the accountability was high. On the other hand, when novel work (high cognitive demand) was done, the flow of classroom activities was unstable, the involvement of students was low, and the accountability was softened. In analyzing data from secondary science, biology, mathematics, social science, and English classes in the Managing Academic Tasks (MAT) study at the University of Texas at Austin, Doyle and his colleagues observed that teachers responded to the pressures caused by novel work in three ways: eliminating the novel work from the class (i.e., avoiding high cognitive demanding tasks); manipulating the credit system; and familiarizing the novel work (i.e., degrading the high cognitive demands). Observing this transformation of novel work into familiar work, Doyle and his colleagues were concerned about the truncation of curriculum in the course of interactions with students.

Building on Doyle's work, researchers in the QUASAR (Quantitative Understanding: Amplifying Student Achievement and Reasoning) project further developed the conceptual framework for analyzing mathematical tasks (Henningsen & Stein, 1997; Stein, Grover, & Henningsen, 1996). Analyzing the task features (i.e., number of solutions, number of representations, and communication requirement) and the cognitive demands of tasks (i.e., memorization, procedures without connection, procedures with connection, and doing mathematics), they found that the task features were consistent, but the cognitive demands were more likely to decline from the task set-up phase to the task implementation phase. By comparing cognitive demands between the task set-up phase and the task implementation phase, they showed that 61 tasks declined their high-level cognitive demands from the set-up phase to the implementation phase, whereas 45 tasks maintained their high-level cognitive demands from the set-up phase to the implementation phase. They further identified factors that contributed to maintaining or declining high-level cognitive demands tasks. Among several factors, shifting the focus to getting the correct answer contributed to declining the high-level cognitive demands of 27 out of 61 tasks (44%), whereas pressing for explanation through teacher questioning, comments, and feedback contributed to maintaining the high-level cognitive demands of 29 out of 45 tasks (64%). These studies showed that eliciting, requesting, and commenting on explanations contribute to maintaining the high-level cognitive demands of mathematical tasks. Analyzing the cognitive demands of mathematical tasks, either during set-up phase or during implementation phase, is not the focus of this dissertation study, but it is important to notice that pressing for explanation serves to maintain the high-level cognitive demands of mathematical task.

The often-used categorizations of mathematical tasks in analyzing curriculum materials or evaluating instructional quality contribute to providing rich opportunities for students to engage in worthwhile mathematical tasks, however, these categorizations might not be successfully applied to reveal different demands of mathematical explanation embedded in different mathematical tasks. Through the iterative data analysis process by using multiple cases of mathematical tasks, this dissertation addresses the need for developing the categorization of mathematical tasks that are aligned with the nature of mathematical explanation.

CHAPTER 3.
METHODS

3.1. Overview

This dissertation aims to conceptualize the work of teaching entailed in supporting students to develop mathematical explanation, particularly using the two key instructional resources—discourse resources and collective resources. As described in Chapter 1, the research questions that guide this dissertation study are:

1. *What are the core tasks of teaching to support students to develop mathematical explanation?*
2. *How are instructional resources used to support the core tasks of teaching?*
    a. *How are discourse resources used to support the core tasks of teaching?*
    b. *How are collective resources used to support the core tasks of teaching?*

To provide an empirical basis for an analytical-conceptual method, I analyzed the data from the Elementary Mathematics Laboratory (EML), a two-week summer mathematics program for entering fifth graders taught by Professor Deborah Ball at the University of Michigan's School of Education, across multiple years. This chapter describes the study design, data source, data analysis, and limitations of this study.

3.2. Study Design

Teaching is often described as a complex activity because it involves managing multiple relationships simultaneously with students and with content over time. To untangle the complexity of teaching, this dissertation adopts an analytical approach that holds other key variables of instruction (the teacher and content) relatively constant, while only varying the students. This section begins with analytical approaches to study teaching and then lays out an appropriate analytical tool that fits the problems under consideration in this dissertation study.

3.2.1. Analytical Methods to Study Teaching

A theory plays an important role in that it provides a lens to explore the phenomenon, aids in the formulation of research questions, contributes to developing explanations of the relationships being studied, and builds the logical structure of the findings (Creswell, 2003; Kerlinger, 1979; Parsons, 1938). Despite its importance, there is a lack of well-developed theories on teaching (Hiebert & Grouws, 2007). Such lack of theories of teaching makes it difficult to articulate an adequate method for studying the distinctive features of teaching. Ball and Hoover (2014) write:

> Instruction is a form of human interaction, but it is a very distinctive form and may call for distinctive methods. In addition, methods for probing these interactions might be drawn from psychology, sociology, anthropology, or elsewhere, but such research may require the development of methods specifically suited for documenting and investigating the distinctive nature of instructional interaction, with its distinctive distribution of responsibility, object of focus for social interaction (viz., a subject area), and goal of effecting student learning. Methods drawn from other disciplines may be better suited for addressing the driving questions of those disciplines, rather than the core problems of instructional practice, and may bring with them native orientations to theory inherent in those disciplines (p. 552).

In many cases, methods for studying teaching have involved adopting terms, concepts, and techniques from other disciplines, but have not further articulated how the selected method addresses issues that matter particularly for teaching. For example, the commonly used qualitative method of grounded theory—originally developed by two sociologists Barney Glaser and Anselm Strauss—has contributed to the construction of a more rigorous theory by adopting the criteria of theoretical sampling, constant comparison analysis and theoretical saturation, but it is not clear how the method itself addresses the distinctive features of teaching. To explicitly make a connection between the phenomenon being studied (i.e., the dynamic interactions) and the method being chosen, this section articulates the study design of this dissertation situated within the instructional triangle (Cohen et al., 2003), which was introduced in Chapter 2.

One way of studying teaching is to contrast the focal components of the instructional triangle under investigation: a teacher, content, students, and environments. First, the contrast between teachers has been used to identify the elements of expertise in teaching (e.g., Leinhardt, 1989; Leinhardt & Greeno, 1986; Livingston & Borko, 1989). For example, Leinhardt and Greeno (1986) contrast the performance of novice teachers to that of expert teachers. By comparing instructional interactions managed by expert teachers with those of novice teachers, they identify that the expert teachers had a larger repertoire of routines, specified activity segments well, and spent less time on guided practice while spending more time on monitored practice.

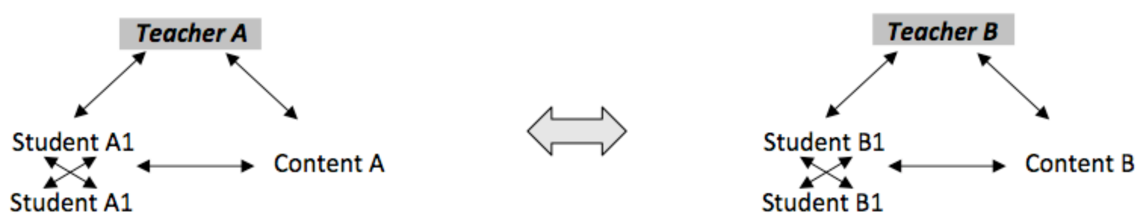


Figure 3.1. A research design that contrast between teachers

Second, the contrast between contents has been used to identify the common model of teaching across disciplines and to specify the distinctive features of each discipline. For example, Leinhardt (2001) identifies the core elements of instructional explanation and argues that the details of each element vary by disciplines. As one of such elements, she illustrates that queries in history play a role of deepening layers of meanings and mismeanings whereas queries in mathematics search for clarification and congruence.

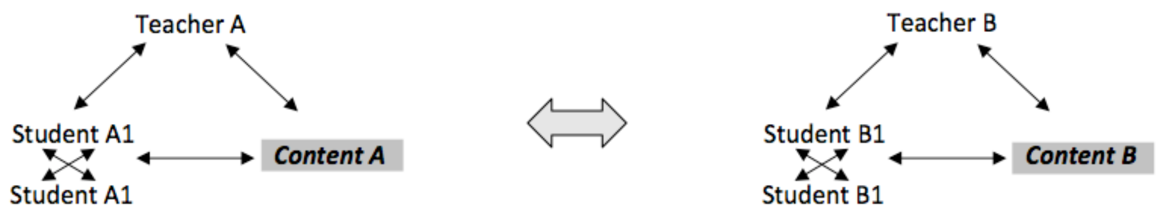


Figure 3.2. A research design that contrasts between contents

Third, the comparison between students has been used to identify how a teacher differentiates instruction according to students' different mathematical abilities. By extending studies on the relationship between the written curriculum and the enacted curriculum, another interest has emerged with respect to how the same teacher enacts the written curriculum differently according to the different level of students' mathematical abilities. For instance, Eisenmann and Even (2009) examined instructional activities provided by one teacher, Sarah, who closely followed the recommendation of the curriculum materials by using the same written curriculum for two different ability groups of students. They found that Sarah did not differentiate the activities between high-achieving students and low-achieving students. Using the same study design, Eisenmann and Even (2011) examined the instructional activities of another teacher, Rebecca, who did not closely follow the recommendation of the curriculum materials for an algebraic activity when teaching two different ability groups of students. Contrary to Sarah's case, Rebecca showed significant differences in activities—more emphasis on meta-level activities such as hypothesizing, justifying, and proving activities for high-achieving students but transformational activities such as substituting, collecting like terms, and factoring for low-achieving students.

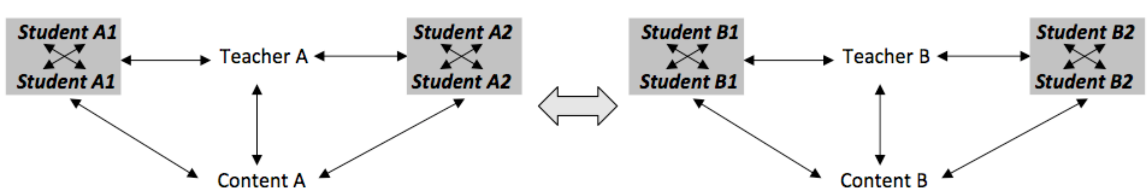


Figure 3.3. A research design that contrasts between students

This dissertation study approaches the complexity of teaching in another way, minimizing much of the variation among components of the instructional triangle in a more controlled context (see Figure 3.4). Analyzing instructional interactions managed by the same teacher teaching the same mathematical tasks to different cohorts of students without substantial differences in students' mathematical abilities is crucial to identify the core tasks of teaching across the particulars of students and unfolding interactions. This method untangles the ways in which the same teacher adjusts instruction to the students' needs in developing mathematical explanation wherein each cohort of students brings different mathematical ideas, stance, issues, language, ambiguity, and difficulties in explaining the same mathematical task. Even though the teacher is held constant, however, there is a nuanced difference related to the content as well as the change in students. Each teaching enactment provides the teacher with deepened knowledge about the teaching of that content, and this may also change the instruction in perhaps subtle ways.

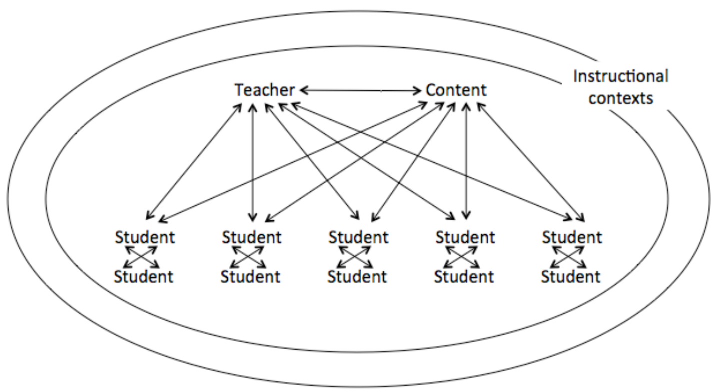


Figure 3.4. A basic research deign which varies students, while holding other variables of instruction constant

Given that one of the greatest predicaments of teaching is its dependence on students (Cohen, 2011), it is important to figure out how instruction might unfold with different groups of students. On one hand, one might imagine that instruction would unfold in the same way by the same teacher teaching the same mathematical task because a teacher might make the same decisions based on his or her knowledge, skills, disposition, and instructional goals. On the other hand, one might imagine that instruction would unfold in a dramatically different way even by the same teacher teaching the same mathematical task because a teacher might make an improvisational decision at the moment. The question of how instruction unfolds with different groups of students may be answered based on one's personal sensibilities or perceptions built through years of their own experiences, but it has not yet been rigorously examined how instruction managed by the same teacher teaching the same mathematical task is likely to unfold differently with different groups of students; how collective resources are likely to be constructed differently with different groups of students; and what is the underlying structure of using collective resources with different groups of students. In other professions, practitioners are trained to adjust the work of doing the same task for different clients. For example, hairdressers have opportunities to learn how to perm different customers' hairs (e.g., based on the thickness of hair or the length of hair) and anesthesiologists have opportunities to learn how to give anesthetics to different patients (e.g., considering age, sex, weight, health conditions, and type of operations). However, as of it is yet largely unknown what is entailed in a teacher's responsible management of instruction in response to different sets of students, their mathematical ideas, and the unfolding of a particular instance of instruction. This dissertation addresses the idea of being responsive to the students' needs by minimizing many of the variations among components of the instructional triangle but only varying the students.

3.2.2. Analytical-Conceptual Method

This dissertation aims to conceptualize the work of teaching entailed in supporting students to develop mathematical explanation, particularly using the two key instructional resources—discourse resources and collective resources. Instead of simply describing what a teacher does, says, or knows and then making judgment about its quality, this dissertation develops a conceptualization about the work of teaching by systematically comparing the emerged themes and testing assumptions grounded on the analysis of empirical data. In analyzing instructional interactions, observers often make statements about what they like or what they do not like; evaluate whether instructions are well aligned with the agendas proposed by professional organizations or state standards; and make suggestions about what a teacher should do or should not do based on their personal experiences or beliefs. Several scholars address this concern:

> Experiences showing video of classroom instruction to people—prospective teachers, education researchers, mathematicians, teachers, policymakers—suggest that people often leap to evaluating teaching, make sweeping claims about what

> the teacher should or should not have done, and talk in general and abstract terms that often reflect vague thinking and are readily misunderstood. Insisting that people refer back to what they heard or saw that led them to react as they did and that they explain how this evidence supports the claim they are making generates more productive commentary and makes both the motives and the substance of ideas more apparent to others. (Thames, 2009, p. 229)

> Such conversations often begin with an observer saying something general about the teaching; usually, “It was terrific” or “It was confusing.” These general comments might be followed by more focused statements like, “You let the boys talk all the time,” “The kind of problem you presented was not realistic,” or “You got into math that was much too complicated for most of the students in the class.” An observer might pick out a particular move I made, noting how I “passed over” Catherine’s disagreement with Richard, how I “diverged from the mathematics” to comment on the socially appropriate nature of the exchange between Ander and Tyrone, how I “imposed” my own interpretation of Ander’s strategy, or how I let the lesson end without “correcting” Ellie’s assertion that 55 divided by 4 was “thirteen point three.” These “descriptions” of what people see in my teaching capture so little of the work as I experience it from inside the role. They judge the practice without analyzing it. And I think, “They just don’t understand.” (Lampert, 2001, p.29)

In analyzing teaching, each person might have his or her own view about what a teacher should do, how instruction should unfold, how instruction matches with a particular learning theory, or how instruction portrays a particular reform idea of teaching mathematics. This dissertation is neither intended to evaluate the quality of mathematics instruction taught by the teacher nor intended to argue that other teachers should do exactly what the teacher did. This dissertation does not aim to merely describe what the teacher does, says, knows, believes, or decides for a particular instructional moment. Rather, this dissertation aims to conceptualize the demands of the work of teaching. In this sense, this dissertation study is not a descriptive study about the teacher, but an analytical-conceptual study which aims to develop a conceptual framework about the work of teaching by systematically testing ideas emerged from the data.

3.3. Data Source

3.3.1. Description of the Elementary Mathematics Laboratory (EML)

To provide an empirical basis for conceptualizing the work of teaching entailed in supporting students to develop mathematical explanation, this dissertation analyzed the data set from the Elementary Mathematics Laboratory (EML), a two-week summer mathematics program for entering fifth graders taught by Professor Deborah Ball at the University of Michigan's School of Education[10], across multiple years[11]. The EML provides opportunities for entering fifth graders to learn several mathematical topics that are important for doing well in the fifth grade (e.g., fractions, writing number sentences, permutations, and number line) and to develop key mathematical practices (e.g., explaining, proving, and using representations). There are no prerequisites to participate in the EML, however it mainly focuses on students who are struggling with learning mathematics rather than students who are outperforming in mathematics. These students are nominated by their classroom teachers, for whom need to develop positive academic identities and to have successful experiences in school mathematics. Considering that the participating students were from the same population—sampled from the same school district—and they were recruited through the same procedure, there were no substantial differences in students' demographics and their mathematical abilities across the five cohorts. Each year, approximately 25-30 students, who are socioeconomically, ethnically, racially, and linguistically diverse, participate in the whole-group mathematics class taught by Deborah Ball in the morning and in the individualized mathematics clinic tutored by undergraduate students in the afternoon. The two-and-a-half-hour long whole-group mathematics class is divided into two sessions with a ten-minute break between the sessions. The first session begins with an individual warm-up problem and the second session ends with an end-of-class check or a refection question. Across these two sessions, the students are engaged in approximately two to four mathematical tasks.

[10] For more information, see http://www.teachingworks.org/training/LaboratoryClasses.
[11] This dissertation analyzes the EML data in 2007, 2008, 2009, 2010, and 2013. No EML class was offered at the University of Michigan's School of Education in 2011 and the EML program was designed for the entering sixth graders in 2012. The EML program was provided at other location in 2011, but excluded from this study because the program was offered in different length (one week) and sampled students from different school districts from another city.

At the same time, the EML also offers opportunities for preservice teachers, inservice teachers, teacher educators, educational researchers, and research mathematicians to study teaching collectively through pre-briefing, classroom observations, and debriefing in the morning as well as participating professional development programs in the afternoon. The staff at the EML documented the videotapes of pre-briefing, mathematics classes, and debriefing as well as lesson plans, students' written work in their notebooks, homework, and quizzes, and observers' comments. The EML has a multi-layered structure, but this dissertation mainly examines the two-and-a-half-hour long mathematics classes taught by Ms. Ball during the two-week program (shaded as gray in Figure 3.5).

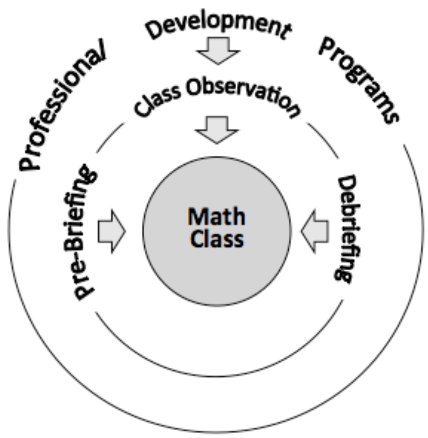


Figure 3.5. A multi-layered structure of the EML program

The lessons were videotaped using two cameras—the mobilized camera (camera A) was located in front of the classroom and the fixed camera (camera B) was located in the back of the classroom. The teacher wore a wireless microphone and students had a fixed microphone in front of their desks. An audio technician monitored the volume of individual student's microphones, so individual students' talk was captured quite well.

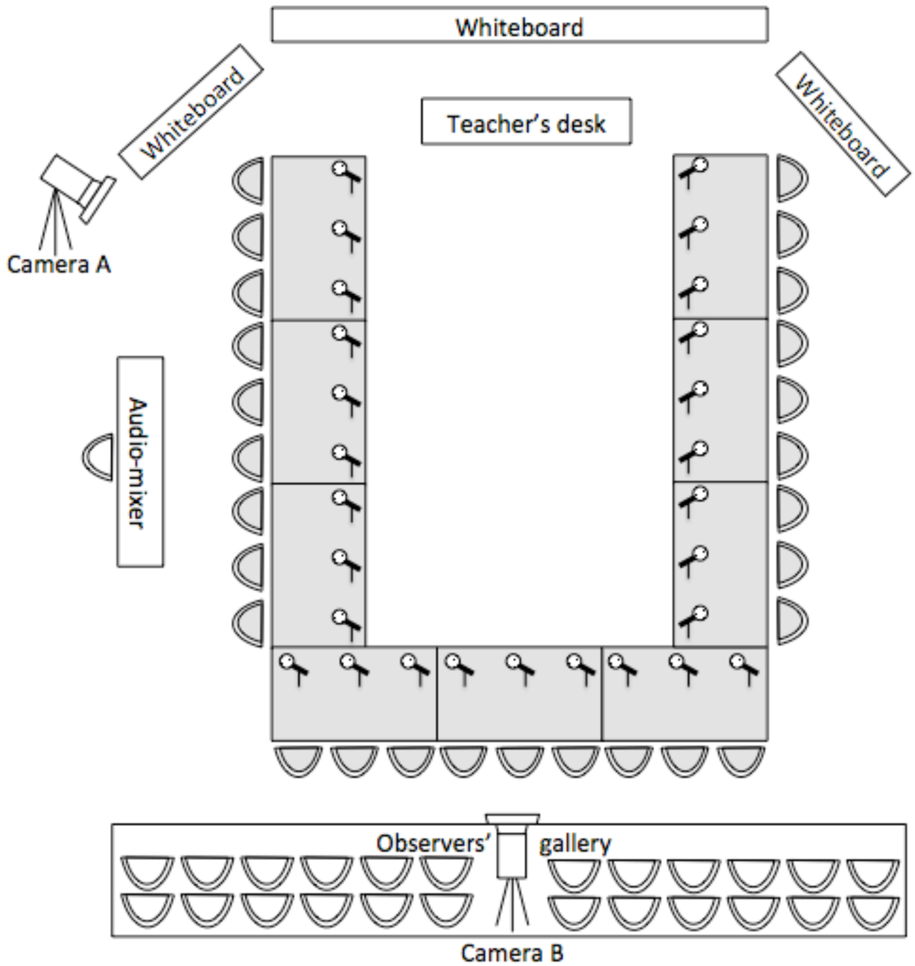


Figure 3.6. The EML classroom arrangement

The data include the video records of mathematics lessons, their corresponding transcripts, lesson plans, and students' written work in their notebooks. The lesson plans provide evidence about the overall structure of the lesson, instructional goals, mathematical territory of the lesson, student learning goals, instructional strategies, and anticipated students' responses. Because the teacher sometimes changed the lesson plan during pre-briefing session based on the EML observers' suggestions, the fidelity to the lesson plan is not the focus of this dissertation study. Figures 3.7, 3.8, 3.9, and 3.10 illustrate the lesson plans for teaching the four mathematical tasks used in the EML 2010.

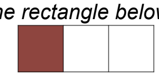
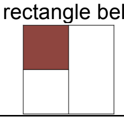
A Lesson Plan for the Brown Rectangle Problem				
Time/ format	*Activity segment*	*Learning goal(s) for students*	*Detail*	*Commentary: notes and anticipations*
10:25-10:50	*Begin work on fractions: Unequally partitioned rectangle problem*	*(G1) Students will begin to understand and use the concepts of "the whole" and "equal parts".* *(G2) Student will begin to use elements of a definition of a fraction to name parts of areas.* *(G3) Students will begin to use the language of the whole and equal parts.*	Problem on chart; ask students to paste it into their notebooks and write what they think: *What fraction of the rectangle below is shaded brown?*  What fraction of the rectangle below is shaded brown?  Use this problem to articulate what is required to explain fractions: identification of the whole, equal partitioning. Try to work with the students to make explicit what students use to construct an explanation of the correct answer. Try to get this to be more explicitly shared: • Refer to the fact that fractions refer to an equally-partitioned whole, not just the number of pieces. • Note that the large rectangle (the whole) is not divided into equal parts. • Divide the large rectangle so that all the parts are equal. • Identify the parts as "fourths" because there are 4 equal parts, because one of the parts iterated four times equals the whole. • Name one of these parts as "one-fourth." Discuss the rectangle with one-third as a contrast, since those who got this question wrong are likely to label it as 1/3.	Begin with an equally partitioned rectangle and use this to contrast and support reasoning about the unequally partitioned rectangle. Students might think that a square is not a rectangle. Will need to discuss/clarify this if it comes up. The task is intentionally drawn with the area of the brown squares in both questions being equal. This allows a discussion of how the fraction of the rectangle is determined by the size of the shaded part in relation to the size of the whole. When the size of the whole changes, the size of the shaded part in relation to the whole changes as well. Students might think that 1/4 of the rectangle is shaded brown when there is a line drawn, but go back to thinking that that same area is 1/3 of the rectangle when you take the line away. Will need to be attentive to this in students' talk. I am not going to define fractions today, but if it seems timely, may work on eliciting key points: • a whole, divided into equal parts • the name of the part is given by the number of those parts it takes to make the whole • one of those parts is called "one-_____ (name of part)". This is the idea toward which we are heading: We will write 1/b, where the whole is divided into b equal parts. a/b is then defined as a • (1/b), or a copies of 1/b. I may not be able to complete this work today, but if possible, I want to be able to get into it to lay the groundwork for the week's learning goals, including being able to find out more about the children's current thinking about this content.

Figure 3.7. A lesson plan for the brown rectangle problem used in the EML 2010

A Lesson Plan for the Blue and Green Rectangle Problem				
Time/ format	*Activity segment*	*Learning goal(s) for students*	*Detail*	*Commentary: notes and anticipations*
9:30-10:15 (partner and whole group)	Blue and Green Rectangle Problem	(G1). Students will understand and use the concepts of “the whole” and “equal parts” to compare, explain, and represent fractions using area model. (B2). Students will able to use a definition of a fraction to identify and name parts of areas.	Discuss learning goals for this segment. Review the two main ideas from yesterday: whole, equal parts (meaning equal areas). Put the following figure on chart paper:  What fraction of the big rectangle is shaded green? What fraction of the big rectangle is shaded blue? Work with partner to figure out the answer. Elicit explanations that use specific key concepts about fractions, focusing on: • identifying the whole equal parts • how many parts • how much is shaded (first green and blue separately and then, if time, together) Elicit point that d copies of 1/d=1 whole. Ask student to consider whether the blue triangle and the green rectangle represent the same fraction of the whole.	Anticipations: Students may argue that blue (or green) part of area is 1/2 of the rectangle by considering the whole to be one-fourth of the entire figure (which is plausible). Try to make clear what the whole is, and in particular, that the question of “what is the whole” is a fundamental fractions question. While students may argue that both the blue and the green parts represent 1/8 of the whole, they may believe that the green part is greater in size than the blue part (or vice versa). We likely will not have time to take up this issue today. Try to get clearer notion of what a “whole” is—i.e., clarify what the “the big rectangle “ is. “Big rectangle” is ambiguous in this diagram. (NOTE: Saying “the whole” in the problem statement is not necessarily clearer, however.)

Figure 3.8. A lesson plan for the blue and green rectangle problem used in the EML 2010

<table>
<tr><th colspan="5">A Lesson Plan for the Two-Coin Problem</th></tr>
<tr><th>Time/ format</th><th>Activity segment</th><th>Learning goal(s) for students</th><th>Detail</th><th>Commentary: notes and anticipations</th></tr>
<tr>
<td>10:40-11:20</td>
<td>Two-Coin Problem</td>
<td>(G6) Student will begin to understand the role of the conditions of a problem and use conditions to reason about solutions.

(G4, G5) Students will begin to learn the importance of being able to ask and answer the question of whether <u>all</u> solutions have been found.

(G7) Student will create records of their solutions and begin to see the need for systematicity.</td>
<td><u>Discuss learning goals for this segment.</u>

Ask a student to read the Two-Coin Problem:
I have pennies, nickels, and dimes in my pocket. If I pull out 2 coins, what amounts of money might I have? Prove that you have found all of the amounts possible.

Ask for one solution. Press students to explain why it does or does not meet the conditions of the problem.
• use only pennies, nickels, and dimes
• uses exactly 2 coins

Pose one or two examples that is not a solution:
• 26¢ with 1 quarter and 1 penny
• 16¢ with 1 penny, 1 nickel, 1 dime
And ask whether these are solutions. Why or why not?

Have students work individually or in pairs.

Elicit and record solutions to the problem into the table.
• When students give amounts, ask them or others what coins could make that amount.
• When students give coins, ask for amounts.

Once a solution is generate, press students to use the conditions to check it.
Ask for MORE amounts after 6 have been found.
When students say there aren't any other solutions, press students to explain how they know that they found ALL of the amounts that meet the conditions:
• Have students talk in pairs: why you are or aren't sure that we've found them all.
• Elicit a few responses.
• Proof of two-coin problem: Develop explanation to prove that all combinations have been found by working with number of pennies (2, 1, 0) and showing all combinations that fit the conditions of the problem.</td>
<td>If time, might ask students to show they have recorded the solutions before showing the table that we will use together. Make comments about the importance of selecting notation and ways to organize solutions.

Solutions:
<table>
<tr><th></th><th>P</th><th>N</th><th>D</th></tr>
<tr><td>2¢</td><td>2</td><td></td><td></td></tr>
<tr><td>6¢</td><td>1</td><td>1</td><td></td></tr>
<tr><td>11¢</td><td>1</td><td></td><td>1</td></tr>
<tr><td>10¢</td><td></td><td>2</td><td></td></tr>
<tr><td>15¢</td><td></td><td>1</td><td>1</td></tr>
<tr><td>20¢</td><td></td><td></td><td>2</td></tr>
</table></td>
</tr>
</table>

Figure 3.9. A lesson plan for the two-coin problem used in the EML 2010

<table>
<tr><th colspan="5">A Lesson Plan for the Three-Permutation Problem</th></tr>
<tr><th>Time/ format</th><th>Activity segment</th><th>Learning goal(s) for students</th><th>Detail</th><th>Commentary: notes and anticipations</th></tr>
<tr><td>9:15-9:25</td><td>Warm up</td><td>(G6) Student will be able to use the conditions of a problem to reason about solutions.
(G7) Students will be able to create and use structures, notation, and other methods to key systematic records of their work.</td><td>How many different three-digit numbers can you make using the digits 1, 2, and 3, and using each digit only once.

Show all the three-digit numbers that you found.

How do you know that you found them all?

Discuss the problem.</td><td>Have students come from snack as they are ready and begin working on the warm-up problem, turn in their homework, etc.

Greet and talk with students individually as they enter and start working.</td></tr>
<tr><td>9:25-9:40</td><td>Discussion of the warm-up problem</td><td>(G4) Students will learn that a claim that you have found all the solutions to a problem is a claim that needs to be proved.
(G5) Students will begin to be able to prove that they have all the solutions to a problem (classification claim).</td><td>Elicit solutions to the problem.

Press students to explain why numbers do or do not meet the conditions of the problem.
• The number has 3 digits.
• The number uses ONLY the digits 1, 2, and 3.
• Each digit is used ONLY once.

Press students to explain how they know that they found ALL of the numbers that meet the conditions.</td><td>Later in the week, students will use permutations of 3 to solve the Train Problem. The goal today is provide students with the experience of finding the exact number of solutions for a permutations of 3 problem.</td></tr>
</table>

Figure 3.10. A lesson plan for the three-permutation problem used in the EML 2010

When analyzing the work entailed in supporting students to develop mathematical explanation, the students' written work in their notebooks provided useful information. First, it provided evidence about how an individual student's mathematical thinking changes, improves, or develops over time through the dynamic interactions with a group of students and the teacher. Second, it provided a basis for comparing verbal explanation and written explanation. What are intrinsic difficulties with producing a written explanation? What are intrinsic difficulties with producing a verbal explanation? How does the production of written explanation contribute to the development of verbal explanation? Even though students produce the same level of written explanation, the ability to provide a verbal explanation might vary by students. Third, the students' notebooks provided the evidence about the scope of students' answers and the proportion of students who produced correct answer to students who produced incorrect answers. Fourth, it provided a basis to examine whether the selection or the sequence of answers relates to the proportion of answers written in the students' notebooks. In the cases where students crossed out their answers or produced multiple answers in their notebooks, however, it is difficult to judge whether they initially came up with those answers or changed their mind during the instruction without an in-depth cognitive interview with those students.

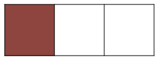
3.3.2. Description of Four Mathematical Tasks

Among the plethora of records of practices of the EML data across five years, I purposefully selected the data that align with the purpose of this dissertation study. After reviewing the EML video indices produced by the EML staff, I searched for mathematical tasks that have been used across multiple years. Among those mathematical tasks, I chose four mathematical tasks[12] that highlight different features of mathematical explanations: (1) the brown rectangle problem (see Figure 3.11); (2) the blue-green rectangle problem (see Figure 3.12); (3) the two-coin problem (see Figure 3.13); and (4) the three-permutation problem (see Figure 3.14).

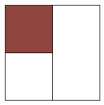
[12] These mathematical tasks are designed by the *Learning Mathematics through Representation* project at the University of California, Berkeley (Geoffrey Saxe and Maryl Gearhart) and by the *Elementary Mathematics Laboratory* project at University of Michigan (Deborah Ball, Laurie Sleep, and Meghan Shaughnessy).

Brown Rectangle Problem

EML2007 What fraction of the big rectangle below is shaded brown?



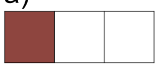
What fraction of the big rectangle below is shaded brown?



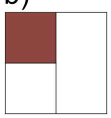
*Note: The problem statements were offered verbally, but not written.

EML2008 What fraction of each rectangle below is shaded?

a)

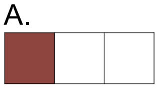


b)

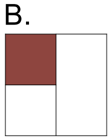


EML2009 What fraction of the rectangle below is shaded in?

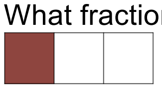
A.



B.

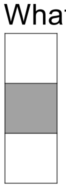


EML2010 What fraction of the rectangle below is shaded brown?



What fraction of the rectangle below is shaded brown?

EML2013 What fraction of the rectangle below is shaded gray?



What fraction of the rectangle below is shaded gray?

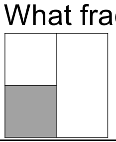
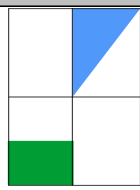


Figure 3.11. The brown rectangle problem

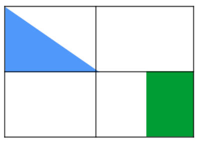
Blue-Green Rectangle Problem

EML2007



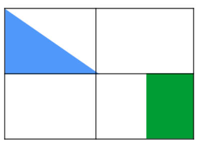
What fraction of the big rectangle is shaded blue?
What fraction of the big rectangle is shaded green?
How much of the big rectangle is shaded all together?

EML2008



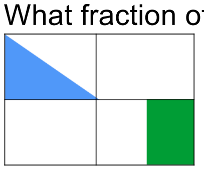
What fraction of the big rectangle is blue?
What fraction of the big rectangle is green?

EML2009



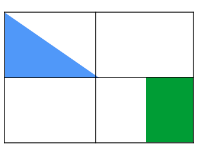
What fraction of the big rectangle is the blue region?
What fraction of the big rectangle is the green region?

EML2010



What fraction of the big rectangle is shaded green?
What fraction of the big rectangle is shaded blue?

EML2013



What fraction of the big rectangle is shaded green?
What fraction of the big rectangle is shaded blue?

Figure 3.12. The blue-green rectangle problem

Two-Coin Problem	
EML2010	[Day03] I have pennies, nickels, and dimes in my pocket. If I pull out 2 coins, what amounts of money might I have? Prove that you have found all of the amount possible. [Day04] Continued from Day03
EML2013	[Day04] I have pennies, nickels, and dimes in my pocket. If I pull out 2 coins, what amounts of money might I have? Prove that you have found all of the amounts possible. [Day10] I have pennies, nickels, and dimes in my pocket. If I pull out 3 coins, what amounts of money might I have? Find all possible answers and prove that you have them all.

Figure 3.13. The two-coin problem

Three-Permutation Problem	
EML2007	[Day08] How many different three-car trains can be made using the light green, purple, yellow rods? Only use one of each. Prove that you have ALL the possible three-car trains?
EML2008	[Day03 Find all the different ways to arrange the r, g, and p (end to end) using exactly one of each rod. How do you know you have made all the possible arrangements?
EML2009	[Day04] Find all the ways to make different trains using exactly one of the red, light green, and purple rods. Keep track of each train. How many are there? How do you know you made all the possible trains using just those 3 rods? [Day05] Three kids ran a race. James, Tasha, and Maria. We don't know the results. We just know that one person finished first, someone finished second, and someone finished third. Make a list of all the possible results of the race. How do you know that you found all of the possibilities? [Day09] Final all the ways to arrange the light green, purple, and yellow rods into three car trains using exactly one of each rod. How are you sure?
EML2010	[Day07] How many different three-digit numbers can you make using the digits 1, 2, and 3, and using each digit only once? Show all the three-digit numbers that you found. How do you know that you found them all? [Day09] Find all the ways to make different trains using exactly one each of the red, light green, and purple rods. Keep track of each train. How many are there? How do you know you have all the possible trains using just those three rods?
EML2013	[Day03] How many different three-digit numbers can you make using the digits 4, 5, and 6, and using each digit only once? Show all the three-digit numbers that you found. How do you know that you found them all? [Day05] How many different ways can you arrange the letters A, B, and C in a row, using each letter exactly once? Show all the arrangements that you found. Explain how you know that you found all the possible arrangements. Underline the conditions of the problem like we did yesterday before you start.

Figure 3.14. The three-permutation problem

Despite holding content (mathematical tasks) relatively constant, there are some minor variations in the layout of the pictorial representation, the inclusion of written problem statement, the wording of the problem statement, the contexts of the problem statement, the presentation of the problem, and the available resources to use. For instance, in the brown rectangle problem, the color of the shaded part, the location of shaded part, the orientation of the rectangle, the wording of the problem statement (e.g., "the rectangle" vs. "the big rectangle"), the availability of sticky lines and sticky brown rectangles vary across years, but the mathematical demands remain quite the same.

As illustrated in Table 3.1, even when teaching the same mathematical tasks, there are variations when they are taught (the early point of the EML program vs. the later point of the EML program), the instructional time within the lesson (at the beginning of the lesson vs. at the end of the lesson), the way of launching (launching as a warm-up problem vs. assigned as homework in advance), and the instructional sequence of similar mathematical problems. For instance, the three-permutation problem was taught after the two-coin problem in the EML 2010, but it was taught before the two-coin problem in the EML 2013. This different instructional sequence can serve to test some hypothesis about whether the three-permutation problem (in which the order of arrangement matters) influences on the explanation for the solutions to the two-coin problem (in which the order of arrangement does not matter) or vice versa. In some cases, the time-constraint, unanticipated students' ideas proposed, or some delays in discussing the previous mathematical tasks in the lesson caused to teach the same mathematical tasks across several days.

Table 3.1. Instructional sequence of four mathematical tasks

	EML2007	EML2008	EML2009	EML2010	EML2013
Brown/gray rectangle problem	Day06 (S1)	Day01 (S2)	Day04 (S1)	Day02 (S2) Day04 (S1)	Day01 (S2) Day02 (B2)
Blue-green rectangle problem	Day07 (S1)	Day03 (S2)	Day05 (S1) Day06 (S1)	Day03 (S1)	Day03 (S2) Day04 (S2)
Two-Coin problem	Day05 (S1)	Day05 (B1)	N/A	Day03 (S2) Day04 (S2)	Day04 (S1) Day10 (S2)
Three-Permutation problems	Day03 (S1)	Day03 (S1)	Day04 (S1) Day05 (S1)	Day09 (S1)	Day03 (S1) Day05 (S1)

Note: S1 denotes the first session of two-and-an-half hour class and S2 denotes the second session of two-and-an-half hour class.

The shaded parts in Table 3.1 represents the data analyzed in this dissertation study. The two-coin problem was mainly introduced as a warm-up problem without the extensive whole-group discussion in the EML 2007 and in the EML 2008, so I did not include the analysis of those two years in this dissertation. Similarly, the permutation problems were mainly introduced as a warm-up problem in the EML 2007 and in EML 2008, so I did not include the analysis of those two years in this dissertation either.

3.3.3. Suitability of the Data

In an attempt to examine the work of teaching entailed in supporting students to develop mathematical explanation, choosing a random instruction does not serve well for the purpose of this study. Rather, this study is a theory-based sampling whereby cases are selected to identify "incidents, slices of life, time periods, or people on the basis of their potential manifestation or representation of important theoretical construct" (Patton, 2002, p.238).

The instruction analyzed in this dissertation has the following four features. First, the instruction is dedicated to cultivating the practice of mathematical explanation. This is explicitly addressed as an instructional goal in the lesson plan and is apparent during observation of the instruction. Without this commitment it would be hard to find evidence of what is entailed in the work of supporting students to develop mathematical explanation. Second, the mathematical tasks are primarily designed to provide rich opportunities to develop mathematical explanation. Thus, it provides an implication for designing mathematical tasks that aim to support students to develop mathematical explanation in classrooms.

There might be concerns about the data analyzed in this dissertation. To further elaborate, I consider the following four issues in this section. First, it might be argued that the data from one expert teacher are not sufficient for understanding the work of teaching because it is difficult to generalize to other settings. However, the purpose of this dissertation is not to examine the status quo of what typical teachers do in supporting students to develop mathematical explanation, but to explore cases in which mathematical explanation is actually manifested.

Second, due to the well-recognized reputation of Deborah Ball as an expert teacher, one might interpret that this dissertation is intended to use her accomplishments as a model for supporting students to develop mathematical explanation. Instead, this dissertation uses evidence from instructional interactions managed by the teacher to support my claims about the core structure of teaching entailed in supporting students to develop mathematical explanation.

Third, due to the multiple instances of teaching the same mathematical task, one might have a tendency to evaluate which approach is the most effective in cultivating the practice of mathematical explanation. For example, in teaching the two-coin problem, the teacher distributed a bag of coin containing pennies, nickels, and dimes to students in the EML2010, whereas she did not distribute it in the EML2013. The purpose of the dissertation is not to compare which approach, either distributing manipulatives or not distributing manipulatives, is more efficient than the other. Rather, the contrast of these two cases serves to elaborate on how students' development of mathematical explanation is influenced by using these mathematical resources (i.e., a bag of coins).

Fourth, because of the chronological features of the EML data, one might assume that this dissertation is a study of teacher learning over time. Due to the nature of public teaching wherein the teacher has often tested several pedagogical or mathematical approaches suggested by EML observers during pre-briefing or debriefing sessions, it is difficult to establish claims about teacher learning. For example, in teaching how to locate fractions on the number line, one of the observers suggested the use of human number line during a pre-briefing session. In this sense, using the particular representation does not necessarily reflect the teacher's knowledge or beliefs. Instead, it shows how different representations provide affordances and challenges for students to explain the mathematical task. In addition, without follow-up interview data, I cannot make claims about what the teacher has learned over time about students and their mathematical ideas, and how she has adjusted her teaching accordingly. Although this dissertation analyzed the data from instructional interactions managed by the same teacher across five consecutive years, it does not make claims about what the teacher has learned over time. Instead, this dissertation examines the core structure and the variations in the instructional interactions across five cohorts, with the variation of students.

Several cautions need to be made about the claims made in this dissertation. First, it is not meant to evaluate teacher knowledge. Even though teacher knowledge is one of the important instructional resources needed to do the work of teaching (c.f., Stylianides, 2007; Sleep, 2009), unpacking the knowledge entailed in doing the work is not a main focus of this dissertation. Second, it is not a study of a teacher's fidelity to the lesson plan or curriculum materials. Even though the lesson plan is provided here, whether the lesson is enacted as planned is not the main focus of this dissertation study. Third, this dissertation study is research on teaching rather than research on teacher education. This dissertation is not meant to explain how teachers learn the practices of supporting students to develop mathematical explanation, but instead it provides implications for that work. The conceptual framework developed in this dissertation is not tested yet in teacher education program or professional development program. Rather, this dissertation is the beginning step for designing teacher education programs or professional development programs that guide learning to teach.

3.4. Data Analysis

This section provides details of the process of analyzing the data in this dissertation study. Due to the large amount of the records of practice, it is important to figure out ways to meaningfully, systematically, and thoroughly reduce the data while maintaining its integrity. To do this, I analyzed the EML data into the following three stages: (1) individual-year analysis (mathematical task fixed); (2) cross-year analysis (mathematical task fixed); and (3) cross-mathematical-task analysis (mathematical task varied). Building on these analytical grounds, I conceptualized the work of teaching entailed in supporting students to develop mathematical explanation into core tasks of teaching and instructional resources. The main goal for each data analysis stage is as follows:

> ***Stage 1. Individual-year analysis***: Identify difficulties that students have and create an initial characterization of mathematical supports provided for students to develop mathematical explanation.
>
> ***Stage 2. Cross-year analysis***: Identify similarities and differences in instructional interactions across multiple years to investigate the relationship between instructional features and supporting students to develop mathematical explanation
>
> ***Stage 3. Cross-mathematical-task analysis***: Identify similarities and differences in instructional interactions across multiple mathematical tasks to generate the ideas around "mathematical generality" and "mathematical specificity" for supporting students to develop mathematical explanation

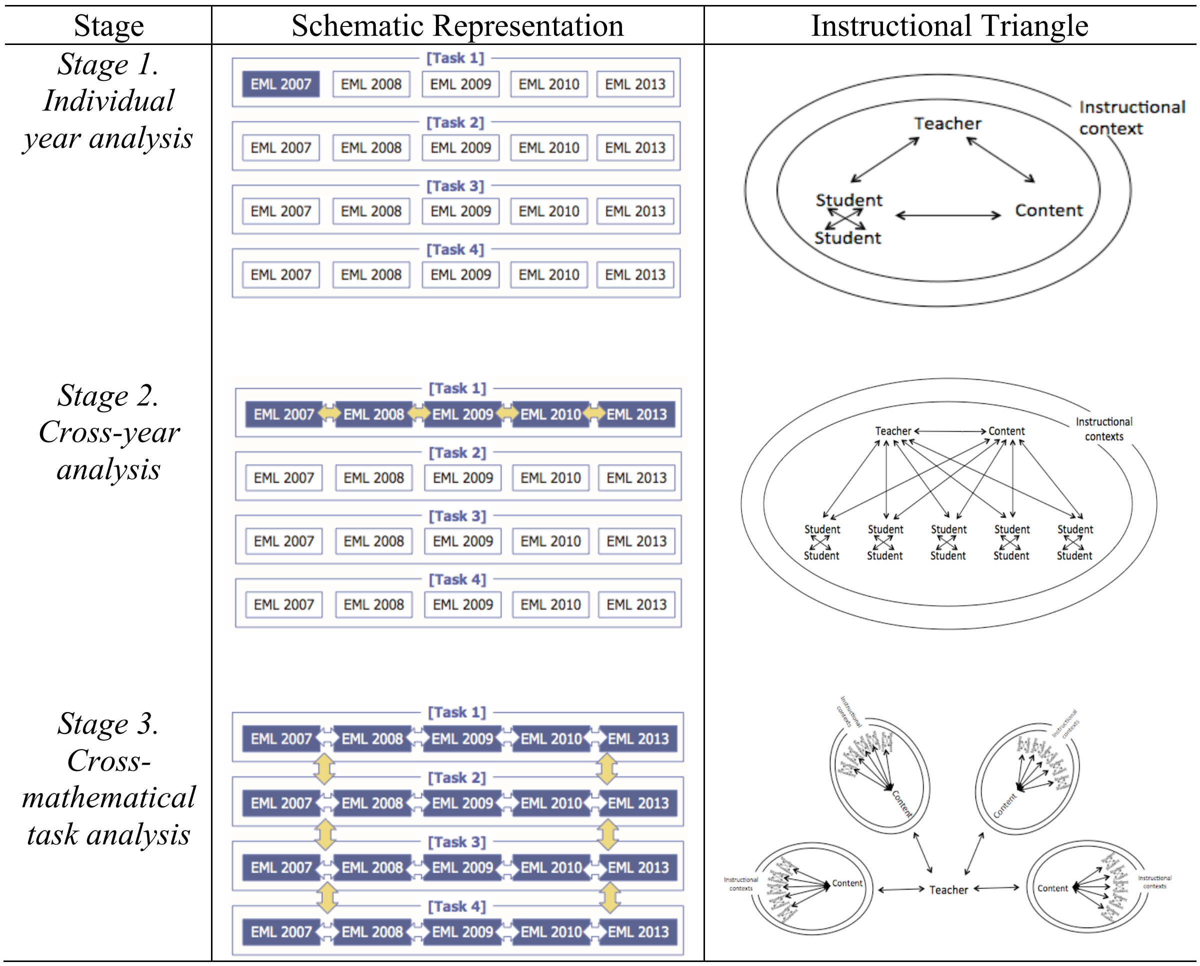


Figure 3.15. Three-stage of data analysis

The first stage of analysis, individual-year analysis, aims to identify difficulties that students have in developing mathematical explanation and to create an initial characterization of mathematical supports provided to students for a particular mathematical task. For this purpose, I analyzed instructional interactions for teaching each individual task in each year by focusing on the key components of instruction. As outlined in Chapter 2, teaching is not just what a teacher says or does, but it involves shaping relationships between groups of students around the content (Ball & Forzani, 2009; Cohen et al., 2003; Cohen, 2011; Lampert, 2001). In doing so, one might think that shaping a relationship with students and shaping a relationship with content are competing, conflicting, and contradicting agendas that are located at the opposite ends in a teacher's mind. Beyond addressing challenges in managing these multiple relationships, Ball and Bass (Ball, 1993; Ball & Bass, 2000) call for developing ways to intertwine bifocal perspectives—mathematical and pedagogical perspectives—that serve the twin imperatives of responsibility to the content and responsiveness to students. In a similar vein, Lampert (2001) argues that teaching involves working on two problems—one of characterizing the subject matter to be taught and the other of characterizing the students to be taught—which have interdependent solutions. Responding to these calls, Thames (2009) specifies ways of coordinating mathematical and pedagogical perspective in analyzing instructional interactions. He writes:

> Part of what is involved in coordinating the twin perspectives of mathematics and pedagogy is that both perspectives need to be engaged and that the thinking and analysis needs to be done across the two. (….) It requires somewhat singular attention to one of the two perspectives, while also listening to and incorporating the other perspective. (p.227)

In line with these ideas, I paid singular attention to each perspective first and then scaled up to how these two perspectives complement, reinforce, and scaffold each other in supporting students to develop mathematical explanation. Figure 3.16 illustrates the specific guiding questions of each perspective. In analyzing the videos, transcripts, lesson plans, and students' notebooks, these analytical questions, in conjunction with literature review, my own teaching experiences as an elementary school teacher, and my research experiences of observing and analyzing teaching in various contexts, guided me to produce the reflective memo that became the foundation of creating an initial characterization of the work of teaching.

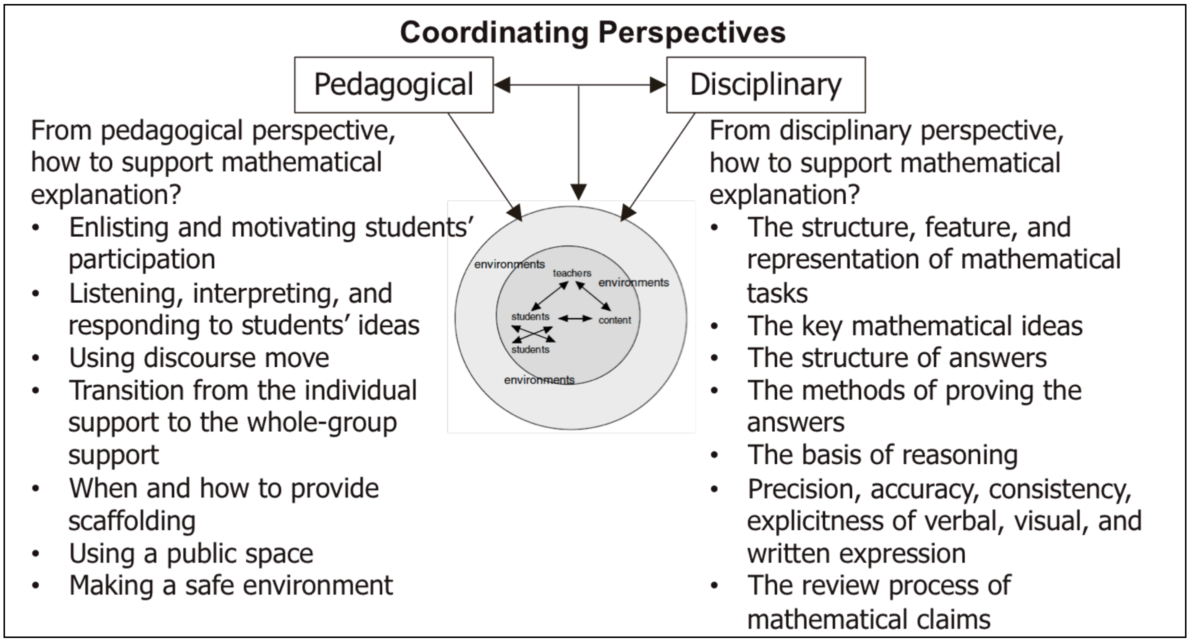


Figure 3.16. Coordinating pedagogical and disciplinary perspectives in analyzing teaching

The second stage of analysis, cross-year analysis, aims to identify similarities and differences in supporting students to develop mathematical explanation by looking at the multiples uses of the same task across years. The similarities across multiple years became strong candidates to be scaled up into the coherent structure of supporting students to develop mathematical explanation, whereas the differences across multiple years offered analytical opportunities to examine whether or not a particular instructional feature plays a role in supporting students to develop mathematical explanation.

The third stage of analysis, cross-mathematical-task analysis, aims to identify similarities and differences in supporting students to develop mathematical explanation by looking at the use of multiple mathematical tasks by the same teacher. The similarities across mathematical tasks serve to generate the ideas around mathematical-task generality in supporting students to develop mathematical explanation, whereas the differences across mathematical tasks serve to generate the ideas around mathematical-task specificity in supporting students to develop mathematical explanation. The three analytical stages I described above are distinguished for an organizational purpose, but do not represent that this is an entirely linear process. Rather, the data analysis is conducted in a more iterative, recursive, and cyclic way.

Building on these analytical grounds, I developed a conceptual framework for the work of teaching entailed in supporting students to develop mathematical explanation. Testing assumptions, hypotheses, and proposals empirically grounded in the specific instances, I asked the following analytical questions to myself in order to decompose the work of teaching into its constituent components:

- What are the possible ways of decomposing the work of teaching? In a sequential structure? In a multi-layered structure? Or in a multi-level structure? What are pros and cons of each approach?
- If the work of teaching is decomposed into multiple hierarchical levels,
    - Is the grain size similar within the same hierarchical level?
    - Is the scale of decomposition from first-level decomposition to second-level decomposition appropriate?
    - Is the work of teaching decomposed into doable and meaningful components? Are the decomposed components too small or too big?
- Does it signify what are mathematically available or attainable for students?
- Does it address mathematical issues or considerations?
- Is it a necessary component to manage difficulties to do the work?
- What are useful strategies or techniques?
- Is it a global (general) idea applied to most of mathematical tasks or a localized (idiosyncratic) idea applied to a particular mathematical task? What does a localized (idiosyncratic) idea tell about the relationship between mathematical task and the work of teaching (or instructional resources)?

Having these rhetorical questions, the initial characterizations, categories, and themes that emerged from individual-year analysis, cross-year analysis, and cross-mathematical-tasks analysis are restructured to accommodate the conceptual underpinnings that serve as the rationale for decomposing the work of teaching in my dissertation.

3.5. Limitations of the Study

This dissertation study has a number of limitations. First, the work of teaching entailed in supporting students to develop mathematical explanation is conceptualized through an iterative data analysis of one expert teacher's responsible management of instructional interactions, so it might not capture the full range of issues that novice teachers struggle with. As Sleep (2009) and Heaton (2000) pointed out in their studies of novice teachers, the practice of novice teachers might be another productive site to identify problems that should be skillfully managed in doing the work of teaching. Given that the deliberate work carried out by an expert teacher is less likely to create problematic situations that interfere with students' engagement in a mathematical task, the data used in this dissertation study might not allow to easily identify such incidents that could be scaled up to the core tasks of teaching. For example, in teaching the two-coin problem in the EML 2010, the teacher distributed a bag of coins, containing the same number of pennies, nickels, and dimes. Such a deliberate preparation of coins creates a physical model for students to come up with all possible combinations of two coins out of pennies, nickels, and dimes. If a drawing is done with replacement, each student should have at least two of each type of coins. On the other hand, if a drawing is done without replacement, each student should have at least four of each type of coins. However, if a teacher was inattentive to the number of each type of coins, such as randomly distributing a handful of coins on the spot without paying attention to the minimum number of each type of coins, a student might not produce all of the possible solutions for the two-coin problem. The data used in this dissertation do not directly provide empirical evidence to identify such problems, but the analytical approach adopted in this dissertation makes me aware of several possible problems that novice teachers might struggle with. It may be ideal to contrast an expert teacher's responsible management with those of novice teachers for teaching the same mathematical task, but it is beyond the scope of this dissertation.

Second, it is evident that students develop their mathematical explanations throughout the EML, but this dissertation does not utilize the results of formative written assessments to quantify the effect of EML on the students' development of mathematical explanation. There are two reasons for this. First, pre-assessment and post-assessment are not always available across all five years, so I did not include those data for the analysis because its availability is inconsistent. Second, more importantly, I notice that the level of sophistication, completeness, and details of students' written explanations are largely influenced by the prompts and contexts of the mathematical task, so it might not serve as accurate evidence to show the students' ability to provide the complete mathematical explanation.

Third, the work entailed in supporting students to develop mathematical explanation is sensitive to language. For example, the use of auxiliary verbs (e.g., "might," "should," or "could") provides an important clue about students' positioning to their mathematical ideas. As another example, the use of particular terms (e.g., "even") complicates the work of supporting students to develop mathematical explanation because its everyday meaning and mathematical meaning are different. The term of "combination" in "combination lock" might be confusing to students because the order of arranging numbers matters for the combination lock, whereas the order of arrangement does not matter for combination in the discipline. The issues around auxiliary verbs or particular terms are important to support students' development of mathematical explanation in the context of teaching mathematics in the U.S., thus these linguistic-sensitive issues might not be important for other countries.

Fourth, the categorization of mathematical tasks identified in this dissertation is not exhaustive. Due to the peculiar context of the laboratory class setting of the two-week program, a limited number of mathematical tasks are available to analyze; among them the aforementioned four mathematical tasks are selected for the analysis. Because of its repeated use across multiple years, it is important to notice that this dissertation illustrates how different mathematical tasks shape the level of mathematical supports and the role of instructional resources rather than intending to propose an exhaustive categorization of mathematical tasks. The findings of this dissertation could be further expanded through analyzing other mathematical tasks.

Lastly, a qualitative methodology involves researchers' interpretation of the data, so the analysis I presented in my dissertation is not free from my personal experiences of teaching mathematics and my research experiences of studying teaching through a number of research projects that I have been involved with.

CHAPTER 4.
CASE 1:
DEVELOPING MATHEMATICAL EXPLANATION FOR THE BROWN RECTANGLE PROBLEM

4.1. Overview

In this chapter, I analyze instructional interactions managed by the same teacher, Ms. Ball, for teaching the brown rectangle problem across five years (EML2007, EML2008, EML2009, EML2010, and EML2013). The brown rectangle problem is composed of two parts: (1) naming a fraction of the equally partitioned rectangle and (2) naming a fraction of the unequally partitioned rectangle. As briefly described in Chapter 3, the brown rectangle problem has been used with slight variations in the layout of the rectangle (where the shaded part is located; the rotation of the drawing), the color of the shaded part (brown, blue, and gray), the inclusion of written problem statements on the poster and in the handout, the presentation of two sub-problems (posting together vs. posting separately), and the wording of the problem statement ("the big rectangle" vs. "the rectangle"; "shaded in" vs. "shaded brown"), but the mathematical demands remain the same across five years. As an area model, the brown rectangle problem reinforces the concept of identifying the whole and making equal parts for naming a fraction in a part-whole relationship.

4.2. The Case of EML 2007

Preview

In the EML 2007, the brown rectangle problem was assigned as homework during the first week and then revisited on the first day of the second week, Day 6. For the first part of the problem (naming a fraction for the equally partitioned rectangle), only one answer (1/3) is proposed by Tatiana. Despite the teacher's attempts to elicit different answers, no further answers are proposed. For the second problem (naming a fraction for the unequally partitioned rectangle), three answers (1/4, 1/3, and 1/2) are proposed by Christopher, Roddie, and Alexa respectively, and one answer (1 and 1/3) is nominated by the teacher based on her observation while circulating the classroom. After listing four proposals on the board, each proposed answer is explained by another student: 1/4 is explained by Lila, 1 and 1/3 is explained by Tatiana, 1/3 is explained by Stan, and 1/2 is explained by Alexa.

After hearing an explanation for each proposal carefully and repeating the initial explanation faithfully, the EML 2007 cohort shifts gears to comparing the reasoning behind 1/4 and 1/3 and then comparing the reasoning behind 1/4 and 1/2. Through the engagement in the comparative analysis of proposals, the EML 2007 cohort collectively drafts two key ideas for naming a fraction—making equal parts and identifying the whole—which ultimately lead to the development of a complete mathematical explanation for the brown rectangle problem. The initial proposal suggested by Tatiana (1 and 1/3) is revised to another proposal (1 and 1/2), another common incorrect answer, so that the revised answer reinforces the idea that identifying the whole is important for naming a fraction even though it names the unshaded parts rather than the shaded parts. An extensive detailed analysis of 30-minute instructional interactions managed by the teacher, Ms. Ball, for teaching the brown rectangle problem in the EML 2007 is provided below.

Extensive Detailed Analysis

After wrapping up a brief discussion about the warm-up problem of Day 6, the teacher makes a transition into the brown rectangle problem. In launching the brown rectangle problem, the teacher shares her observation that many students got the same answer for the first part of the problem but had different answers for the second part of the problem in the last week's homework. The teacher posts the first problem on the board and then asks students to write down "what fraction of *the big rectangle* is shaded brown?" in their notebooks. The problem statement is not written on the poster, but verbally provided by the teacher. At this point, the teacher does not specify what *the big rectangle* refers to at this point. Even though the students already solved the brown rectangle problem as homework in the previous week, they get another chance to work on the brown rectangle problem on Day 6.

While the students are working on the problem individually, the teacher circulates the classroom to check students' work, but she does not provide any substantial mathematical supports. After a minute later, the teacher reconvenes the class to discuss what number the students wrote and why.

In responding to the teacher's call, Tatiana makes the proposal of 1/3 and then explains, "Because there's three squares and one of them is shaded." After Tatiana offers an explanation, the teacher invites Tatiana to the board to write her answer. Tatiana accepts the invitation without any hesitation and makes her way to the board. Meanwhile, the teacher asks someone to repeat why Tatiana picks 1/3 for the first problem. Niena repeats, "Because there's one shaded and three squares." Until this point, the teacher neither evaluates the correctness of the answer nor fills in the missing information in the given explanation. Instead, the teacher repeats Tatiana's initial explanation, "There's three squares and one of them is shaded," and then asks whether anyone has different numbers than 1/3.

The teacher does not explicitly ask "Do you agree or disagree?" with the correct answer that Tatiana proposes, which is often considered as a "must-be-used" catalyst to facilitate a whole group discussion, but makes attempts to elicit different answers. The teacher is aware that everyone gets 1/3 for the first problem—both from analyzing homework and from scanning the notebooks while circulating the classroom during individual work—, but makes space for students to draft other possible answers on the spot and creates a context to reach a tacit collective agreement about the answer. As nobody proposes different numbers than 1/3 for the first problem, the teacher reiterates her initial observation that she does not think that the class disagrees with the answer for the first problem and then moves onto the second problem. The EML 2007 cohort reaches a tacit collective agreement about the answer of 1/3 for the first problem right away without any further objection. The absence of incorrect answers—mainly by mathematical design of the first problem—does not create an opportunity to elaborate the incomplete explanation given by Tatiana.

The teacher then posts the second problem on the board. For the second problem, the problem statement is not written on the poster but verbally provided by the teacher. The teacher asks students to write down what fraction of *the big rectangle* is shaded brown in their notebooks. Like the first problem, the teacher does not clarify further what *the big rectangle* refers to at this point. The referent of “the big rectangle” might not be confusing for students to solve the first problem, but might be interpreted differently in solving the second problem, depending on the level of their understanding about the inclusive relationship of quadrilaterals. If someone does not establish the concept that a square is a special kind of rectangle, one might choose the left side of the whole rectangle (i.e., a non-square rectangle) as the big rectangle than choosing the whole rectangle (i.e., a square).

After posting the second problem, the teacher reminds the students that she observed very different answers about the second problem from the last week’s homework. The lesson plan provides the information that six students (Daniel, Jaffa, Lila, Marcel, Ethan, and Micah) got 1/4 (correct answer), 21 students got 1/3 (incorrect answer), and two students got 1/2 (incorrect answer)[13]. It is possible that some students might have changed their mind over the weekend or during the lesson through an interaction with other students, thus this information might not be entirely transferable to use on Day 6.

[13] The teacher left a question mark for the remaining four students. It is unknown whether these four students did not make any records on the homework or they crossed out the answers so the answers are not recognizable.

Assigning the brown rectangle problem as homework and carefully examining the answers that students produced in advance, however, allows a teacher to have an approximate anticipation about mathematical ideas that the EML 2007 students bring to a whole-group discussion. Given that teaching depends on its students, the work of supporting students' development of mathematical explanation might entail customizing prompts and questions according to the mathematical stance possessed by students. More specifically, the prompts and questions to support students' development of mathematical explanation might be different when most students have the same mathematical idea and when there is a lot of tension between two competing claims. Even in the case where a majority of students come up with the same idea, the prompts and questions might be different when they come up with a correct answer than when they come up with an incorrect answer. If this is the case, preliminarily surveying the answers in this manner reduces the demand of figuring out the composition of students' mathematical stance rigorously during the lesson.

During individual work, the teacher circulates the classroom but mostly checks whether everyone has written down a number in his or her notebook. She then shares her new observation about the answers for the second problem, repeats Tatiana's initial explanation, and then launches a whole-group discussion. The teacher says:

> Okay, there are about three different answers[14] in the notebooks when I'm walking around. So we're going to see if we can talk this through and agree on one answer. Because there aren't three completely different numbers that would describe what fraction of this rectangle is shaded. So when we hear different people's reasoning, you have to think very carefully about how you can explain. When Tatiana explained this, she said there are three squares and one of them is shaded, and everyone agreed. And we didn't really talk very much about whether that was a good enough explanation, because you all agreed with her. So, now we're gonna talk about this one [pointing to the second problem] and listen to people's explanations, and then we still might go back to this one [pointing to the first problem] and see if the explanation was complete or not.

[14] The teacher made a note that she observed 1/3, 1/4, and 1/2 from the previous week's homework in the lesson plan, but observed 1/3, 1/4, and 1 and 1/3 during the lesson of Day 6. It is possible that two students who wrote 1/2 in the homework changed their answers during the lesson, maybe being influenced by other students.

Providing these comments allows the teacher to allude the idea that agreement plays a somewhat contradictory role in the development of mathematical explanation: reaching a collective agreement is an important mathematical goal to achieve in the end, but reaching a unanimous agreement at the very early stage creates a missed opportunity for students to scrutinize an incompleteness of given explanation and pre-empts the need to develop a complete mathematical explanation, especially if the answer is correct.

In addition, the teacher legitimizes disagreement about the answer for the second problem and places a mathematical value on disagreement, not just using it as a driving force to increase the amount of students' talk. The process of resolving disagreement about the answers for the second problem reveals an important mathematical idea, relationship, and structure: mutually incompatible answers reveal disparate mathematical reasoning behind an answer; if answers are not mutually incompatible, they are somewhat mathematically interconnected[15]. In addition, this process provides a tool for students to check whether the unanimously agreed-on explanation for the first problem is complete or not.

In responding to the call for proposals for the answers of the second problem, Christopher shares his answer of 1/4. Instead of scrutinizing each proposal one by one, the teacher continues to collect other proposals and makes a record of them on the board. Roddie adds 1/3 to the list and Alexa adds 1/2 to the list. Although the teacher did not observe 1/2 in the students' notebooks while circulating the classroom[16], she adds Alexa's proposal to the list. Alexa wrote 1/4 in her notebook, but told that somebody had 1/2. Alexa might observe that Amber, who was sitting next to her, wrote 1/2 for the second problem in the notebook. The teacher then adds 1 and 1/3 to the list in which she observed while circulating the classroom. After listing these four proposals on the board, the teacher asks students to explain each proposal.

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[15] In case of the second part of the brown rectangle problem, 1/3 and 1/4 are mutually exclusive answers, whereas 1/4 and 2/8 are not mutually exclusive answers. Mutually exclusive answers (e.g., 1/3 and 1/4) reveal disparate mathematical reasoning for naming a fraction—1/3 does not consider equal parts whereas 1/4 considers equal parts. Not mutually exclusive answers (e.g., 1/4 and 2/8) reveal a mathematically interconnected concept—equivalent fraction.

[16] In the lesson plan, the teacher specified that two students wrote 1/2 for the second problem but did not observe 1/2 while she was circulating the classroom.

The review of the four proposals begins with Lila's explanation of 1/4, which was initially proposed by Christopher. Accepting the teacher's invitation to the board, Lila goes to the board and then explains:

> Well, you see, it can't be one-third 'cause they're (pointing to the right side of the big rectangle, the left lower side of the big rectangle, and the left upper side of the big rectangle) not all equal, but if you split this rectangle in half, (drawing an additional horizontal line on the right side of rectangle) it'll become one-fourth. 'Cause there will be four squares inside the one square, and one square will be shaded.

In beginning her explanation, Lila valorously disputes the proposal made by Roddie, which is an incorrect but a dominant answer that most of her classmates produced in their notebooks (she might not be aware of it though). In refuting 1/3, Lila supports her claim by pointing out that the parts are not all equal, which serves a reason for drawing a line and calling it 1/4. Lila's language is inaccurate ("in half" to indicate that she divides the right side of the big rectangle into the two equal squares), unclear (whether "equal" means "equal shape" or "equal area"), ambiguous (labeling "one square" to indicate both for the whole and for the shaded part) and incomplete (missing "equal" in her final explanation), but she accurately uses geometric names (square and rectangle) and thoroughly provides the details of her reasoning.

More importantly, Lila proffers the idea of "equal" which is key for developing a complete mathematical explanation for the brown rectangle problem. Although the term "equal" is what the teacher strives for, aims to elicit, and hopes that students have proficiency in using it, she does not take up this idea immediately at this point. Nor did the teacher rigorously inspect the inaccurate, unclear, ambiguous, and incomplete aspects of Lila's explanation. Avoiding the evaluation of Lila's initial explanation, the teacher asks whether anyone has a question for Lila. As Lila disputes the proposal of 1/3, which most of the EML 2007 students produced in their notebooks, eliciting a question for Lila at this point functions as making a direct connection to the proposal of 1/3. Most of the EML 2007 students produce 1/3, but no one challenges Lila. The teacher does not rush into confronting these two proposals, 1/4 and 1/3, but asks Lila to repeat why it cannot be 1/3. Lila repeats her initial explanation:

> It can't be one-third 'cause this [pointing to the right side of the big rectangle] used to be a big rectangle, and it wouldn't be even 'cause these are two squares. So I, if you just cut this one in half so it makes two more squares, there'll be four squares. And one would be shaded, so it'll be one-fourth.

In repeating her explanation, the mathematical ground of her explanation remains quite the same. However, in adding more details, Lila increases inaccuracy in her repeated explanation such as replacing the word "equal" with "even" and implying that "unevenness" is attributed to different shapes—the big rectangle and two squares—rather than different size. Additionally, her repeated explanation adds more complexity because she names the right side of the rectangle as a big rectangle, whereas the teacher labels the whole rectangle as a big rectangle. The teacher neither problematizes the increased inaccuracy and more complexity in Lila's repeated explanation nor asks "Do you agree or disagree?" with Lila's answer to the class, which is often considered as a natural discourse move a teacher would be expected to employ. Given that four proposals are already made in a public space, employing the discourse move of "Do you agree or disagree?" might ignite a vigorous debate about answers. However, asking for agreement about the correct answer at the very early stage closes a door to listen to reason about incorrect answers, which plays an important role in the development of mathematical explanation for the brown rectangle problem.

The teacher then moves onto another proposal. Tatiana volunteers to explain 1 and 1/3 but Marcel, who wrote 1/4 both in the last week's homework and in his notebook during the lesson, jumps in and asks for someone else to explain 1/4. Because of the lack of accuracy, clarity, and explicitness in Lila's explanation, it is quite reasonable for Marcel to request another explanation for 1/4. The teacher postpones Marcel's request until all proposals are reviewed. Delaying Marcel's request might indicate that the EML2007 cohort does not develop available and sufficient resources to elaborate inaccurate, unclear, and incomplete but very detailed explanation Lila gave at this point. Thus, eliciting another explanation from another student would not make a significant contribution to the development of complete mathematical explanation for the brown rectangle problem at this point. It is noticeable that the teacher mainly elicits one explanation for each proposed answer but makes use of the initial explanation thoroughly by repeating, revoicing, asking questions, and making further comments rather than eliciting various explanations for the same answer. One might think that it is ideal for a teacher to elicit a number of different versions of explanations for the same answer by distributing turns to as many students as possible. In doing so, various explanations for the same answer are held in reserve for the development of mathematical explanation, which has the benefits of offering selectivity, richness, thoroughness, efficiency, and effectiveness, but also might increase the complexity a teacher faces when inspecting each explanation rigorously and managing the relationship among various explanations at the same time.

After postponing Marcel's request, the teacher gives another turn to Tatiana to explain the proposal of 1 and 1/3:

> Well, it's one and one-third, I think, because this [pointing to the right side of the big rectangle] is one, and this [pointing to the left-bottom piece of the big rectangle] is a half, these two, so this [pointing to the left-bottom piece of the big rectangle] is one right here, so all these are three sides, so that'll be one and one third all together.

Tatiana's explanation is both mathematically and pedagogically challenging to decipher. Unlike the clear explanation that she gave for the first problem, Tatiana hastily points to the pieces by naming all of them with a demonstrative pronoun (i.e., this) rather than moving her finger around the piece and naming it with geometric names. In addition, it is unclear what "a half" refers to and how it is counted for naming the fraction as 1 and 1/3. Along with the incorrect use of mathematical term of "three sides," Tatiana draws two different referents in naming a fraction: a half of the big rectangle as a whole for naming the whole number part (i.e., 1) and the big rectangle as a whole for naming the fractional part (i.e., 1/3). Tatiana names both the right side of the rectangle and the left-bottom piece of the rectangle as one. In addition, she does not consider "equal" for counting parts.

Thames (2009) describes this situation as "mathematically motivated listening that attends to the words and ideas being expressed in a framework of the interplay between ambiguity and precision." (p.157). Facing the demanding work of interpreting Tatiana's explanation, the teacher asks Tatiana to explain again, while seeking very careful attention from other students. Tatiana repeats:

> It's three pieces so that's a third. And this (pointing to the right side of the big rectangle) is one, so that would be one. And that's another half of that so it'll be one and one-third.

Staying with the same logic, Tatiana repeats her initial explanation by changing the order of information: explaining the number of pieces first and then indicating a referent of the whole. Unlike Lila who increased inaccuracy in her repeated explanation, Tatiana reduces inaccuracy a little bit in her repeated explanation by substituting the word "three sides" with "three pieces." However, she still uses a demonstrative pronoun only (i.e., "this") instead of using a geometric name (i.e., this little rectangle) and remains unclear how "half" is used for naming the fraction as 1 and 1/3. It is interesting to observe that the way Tatiana uses the term "half" (which indicates another whole) in explaining 1 and 1/3 is different from the way Lila uses the term "half" (which indicates being divided into two equal parts) in explaining 1/4. The teacher gets a confirmation from Tatiana about her answer, 1 and 1/3, and then invites other students to ask a question for Tatiana. At this moment, the teacher emphasizes that the point is not to argue with Tatiana but asks other students to understand Tatiana's thinking. In responding to the teacher's request to explain what Tatiana said, Hilaire explains:

> Because she's counting all the pieces together and then she's counting the whole, which is one. And then all the pieces is three and then there's one shaded, so she has one and one-third.

Hilaire does not specify whether Tatiana counts the big rectangle as a whole or the right side of the big rectangle as a whole, but marvelously deciphers Tatiana's challenging explanation. He uses the same language that Tatiana chose (e.g., pieces) but dramatically eliminates some ambiguity embedded in Tatiana's explanation. In addition, he decomposes where the whole number part (1) comes from and where the fractional part (1/3) comes from. After checking back with Tatiana whether Hilaire has captured her thinking, the teacher praises Hilaire's nice listening and moves onto another proposal.

Upon the teacher's request, Stan explains 1/3, which was originally proposed by Roddie. Although Lila refutes 1/3 at the beginning of a whole-group discussion, the teacher creates an opportunity to defend the proposal of 1/3. Stan explains:

> I think that it's one-third because one shaded in out of three parts, two of those (pointing to the unshaded parts), and one shaded, so it's one-third.

There are some mathematical similarities and differences between Stan's explanation for his incorrect answer of 1/3 for the second problem ("one shaded in out of three parts") and Tatiana's explanation for her correct answer of 1/3 for the first problem ("three squares and one of them is shaded"). Both do not pay enough attention to the equal parts but just count the number of parts. In a closer examination, these seemingly similar mathematical explanations can be distinguishable. Stan chooses the language of "parts" in explaining his incorrect answer for the second problem, but Tatiana uses the language of "squares" in explaining her correct answer for the first problem. It is interesting to observe that Tatiana chooses the language of "three squares" in explaining her correct answer of 1/3 for the first problem but she chooses the language of "three pieces" in explaining her incorrect answer of 1 and 1/3 for the second problem.

It is not known whether such a language choice might be a personal preference or impromptu and whether there exists any pattern or consistency with using language in naming a fraction, but a brief review of the language that Stan, Tatiana, and Lila chose reveals mathematical issues that students might be grappling with, either consciously or unconsciously. It does not seem to matter whether students choose "squares" or "parts" in explaining the correct answer both for the first problem and for the second problem, but this choice seems to reveal a mathematical issue with explaining an incorrect answer of 1/3 for the second problem. In explaining the incorrect answer of 1/3 for the second problem, naming it as "three squares" violates the definition of a square (because the right side of the big rectangle is not a square but a rectangle) and naming "three rectangles" presumes understanding about the inclusive relationship between quadrilaterals. In this sense, it might be more natural for the students who have the incorrect answer of 1/3 for the second problem to choose "three parts" or "three pieces" rather than "three squares" or "three rectangles" in explaining the incorrect answer.

Stan's choice of "three parts" instead of "three squares" or "three rectangles" avoids confronting the above-mentioned geometrical issues. Again, neither asking "Do you agree or disagree?" nor asking for further comments on Stan's explanation, the teacher asks someone to repeat Stan's explanation. Micah, who wrote 1/4 both in the homework and during the lesson, explains:

> He is counting every single piece of that, of this, of the square as one-third. The shaded in is the only third in there.

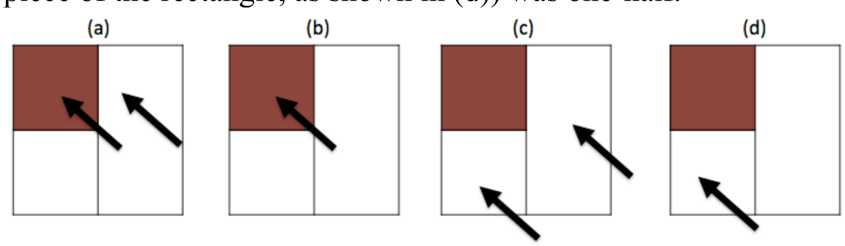
It might be challenging to detect the subtle difference between Stan's explanation and Micah's explanation. On closer examination of these two explanations, it is clear that Micah's explanation provides information about counting each piece as 1/3 rather than the information about how to name the shaded piece as 1/3. Micah's explanation could be used to challenge Stan's answer by requesting Stan to explain how to name each of three pieces. Stan obviously names the left-upper piece of the whole rectangle as 1/3 but does not make explicit that he also considers the right side of the whole rectangle as 1/3. In addition, Micah makes the idea complicated by saying that Scot counts every single piece as 1/3 at the beginning but saying that the shaded piece is the "only" third at the end. It is not clear what the intended meaning of "only" in Micah's explanation is. In the context where Micah provides his own interpretation rather than repeating Stan's explanation as the teacher requested, detecting such a subtle difference, the teacher asks Micah to explain how Stan names 1/3 again, but he could not provide a further explanation.

The teacher asks someone to explain why Stan counts it as 1/3 and Mahluli explains "Because one is shaded in and there's three squares." Stan initially explains that one out of "three parts" is shaded, but Mahluli explains that one out of "three squares" is shaded. Mahluli captures the mathematical structure of Stan's explanation relatively well, but does not use the term correctly because the three parts are actually composed of two squares and one rectangle instead of three squares. The teacher does not further problematize Mahluli's incorrect use of a geometrical name at this point, but praises his good listening.

Pointing out the incorrect uses of a geometric name allows students to be aware that the second problem is not composed of three squares but two squares and one rectangle or three rectangles (two small rectangles and a little bit bigger one rectangle), which leads to the development of the idea of "equal," but this could distract students' attention to the inclusive relationship of quadrilaterals (i.e., a square is a special kind of a rectangle) for a while. Even if the teacher corrects "three squares" to "two squares and one rectangle," the work of supporting the development of mathematical explanation is still complex because just providing different geometrical names is insufficient to prove that the parts are not equal.

The teacher then seeks for an explanation for the last proposal which is not explained yet. Two other students, Kurtis and Roddie raise their hands, but Alexa gets the floor to explain her reasoning on the board. It is not clear what they came up with the answer for the second problem because they completely crossed out something in their notebooks, but none of them produced 1/2 as an answer for the second problem in their notebooks. Alexa accepts the invitation to the board and gives an explanation.

Alexa: I think that they said one-half because two rectangles (pointing to the left side of the big rectangle and the right side of the big rectangle, as shown in (a)) and they only count this (pointing to the left-upper side of the big rectangle, as shown in (b)) as one, and they count both rectangles (pointing to the left-bottom of the big rectangle and the right side of the big rectangle, as shown in (c)) together. So they thought that this (pointing to the left-bottom piece of the rectangle, as shown in (d)) was one-half.

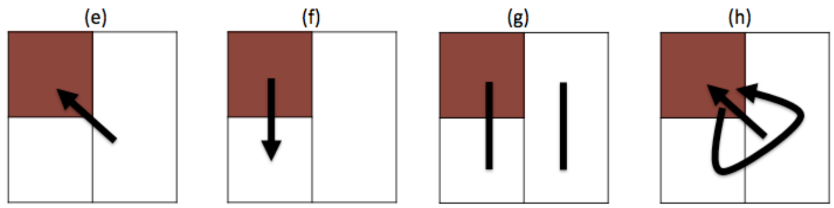


Teacher: I think that you're not saying something. Say a little slow that people can understand you.

Alexa: I think they thought one-half because it's two rectangles (pointing the left side of the rectangle and the right side of rectangle with a marker)

Teacher: Where are two rectangles? Can you show people where're two rectangles?

Alexa: One here (pointing to the left side of the big rectangle), and one here (pointing to the right side of the big rectangle). They would just cross the line, just like that, they count one of them (pointing to the left-upper piece of the rectangle, as shown in (e)) is shaded in, and put these two blocks together (pointing to the left side of the big rectangle, as shown in (f)) and then made the regular rectangle, and there's two (pointing the right side of the rectangle and the left side of the rectangle, as shown in (g)), so they say one-half (pointing the left-upper piece of the rectangle and then circling the inside of the rectangle, as shown in (h)).



Teacher: And ignore the other one?

Alexa: No. This one (pointing to the right side of the rectangle)?

Teacher: Ignore the other one? Ignore that one? Didn't pay attention to it?

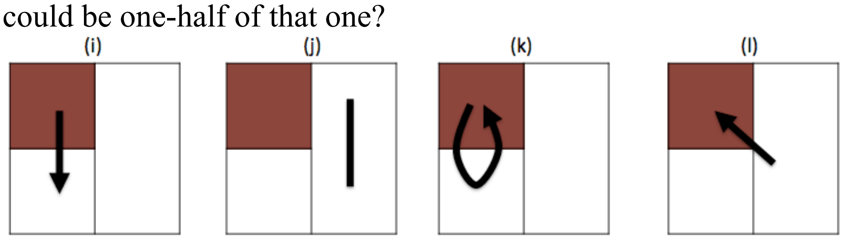
Alexa: They didn't pay attention to it.

Teacher: Okay, did people understand what Alexa is saying?

Students: No.

Students: Yes.

Teacher: So, this student thought this is two rectangles (pointing to the left side of the big rectangle, as shown in (i)) and didn't pay attention to this one then (pointing to the right side of the big rectangle, as shown in (j)), and looked this right here (pointing to the left side of the big rectangle, as shown in (k)), and do you see how this (pointing to the left-upper side of the big rectangle, as shown in (l)) could be one-half of that one?



Like Tatiana's explanation for her incorrect answer of 1 and 1/3, Alexa's explanation is mathematically and pedagogically challenging to unpack what it exactly means. Alexa explains that she only counts the left side of the big rectangle as one, but it is not clear whether she counts it as one whole or as one shaded part. In addition, as shown in (a)-(d), she hastily points to the parts, sometimes for the left-upper piece of the big rectangle as shown in (a) and (b) and sometimes for the left-bottom piece of the big rectangle as shown in (c) and (d). Thus, it is not clear whether she merges the left-upper piece of the big rectangle and the left-bottom piece of the big rectangle together as one shaded part but keeps the big rectangle as a whole (as shown in Figure 4.1) or she keeps the shaded part but considers the left side of the big rectangle as a new whole in her initial explanation (as shown in Figure 4.2).

Figure 4.1. Keeping the size of whole but changing the portion of shaded part

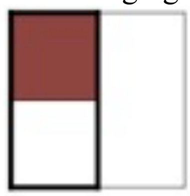


Figure 4.2. Keeping the shaded part but changing the size of the whole

After Lila's initial explanation, the teacher points out that something is missing in Lila's explanation and asks her to explain again. When Alexa provides her second explanation but still hastily points out the pieces, the teacher pauses the explanation to clarify which rectangles Alexa refer to. In her repeated explanation, Alexa adds that she put two blocks together (the left-upper piece of the big rectangle and the left-bottom piece of the rectangle) to make a regular rectangle in which was not explicitly addressed in her initial explanation. It seems that Alexa intends to indicate a "regular rectangle" as an oblong, a non-square rectangle, but her language contradicts with the mathematical definition of a "regular rectangle" which means a square. In addition, it is not clear whether making a "regular rectangle" means enlarging the shaded square into the shaded oblong as shown in Figure 4.1 or shrinking the square whole into the oblong whole as shown in Figure 4.2.

Due to the double-barreled interpretation about 1/2, the teacher points out the missing information in Alexa's initial explanation directly, intervenes to clarify what two rectangles she refers to and what she ignores, and revoices Alexa's explanation rather than using discourse moves that the teacher previously used, such as requesting for repeating or inviting other students to ask questions. The teacher praises Alexa's effort to explain someone else's thinking.

Shifting gears from hearing an explanation of each proposal to making a connection between proposals, the teacher reminds the students not to disagree but to make efforts to understand each other's thinking. The teacher clarifies the shared understanding about naming a fraction—counting the parts—and then prompts students to think about the differences between proposals. Christopher, who initially proposed 1/4, takes up the floor to defend his answer of 1/4 rather than explaining the different reasoning between proposals. He explains:

> I think it's one-fourth because, 'cause they're just trying to trick you. 'Cause even if, even, you don't even ha, you don't have to have any lines, it's still one-fourth of the box.

The teacher prompts students to compare the difference between 1/3 and 1/4 in a neutral tone, but Christopher defends his proposal of 1/4. Christopher had an incorrect answer in the last week's homework, but wrote 1/4 in his notebook during the lesson. It seems that he, himself, thought that he was tricked by the problem when he did the homework. Christopher asserts that it is still 1/4 without having any lines, but neither visualizes his idea in a public space nor provides supportive mathematical evidence of his claim. Christopher does not directly address the teacher's question, but the teacher tries to expand on his idea. When the teacher asks other students to explain why the problem is tricky, Marcel, who got the correct answer both for the last week's homework and during the class, explains:

> It's trying to trick you by making you think that it's, uhm, that, that you don't see that it's fourth because the line is not there, but it doesn't really matter 'cause it's equal to.

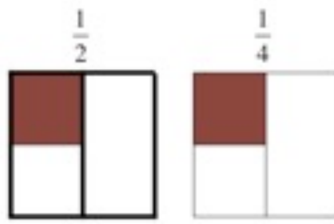
Marcel sheds light on the idea of "equal" which Lila proffered at the very beginning stage and adds mathematical evidence of the claim that Christopher offered. Both Christopher and Marcel argue that drawing a line does not matter for naming a fraction to support the proposal of 1/4, whereas drawing a line is an important mathematical tool that helps students see four equal parts, and thus needs to be highlighted. The teacher does not diverge into the issue of whether or not drawing a line matters for naming a fraction, but redirects students' attention to why it is important to fill in the line as Lila did and asks what kinds of parts are important for naming a fraction. Amber suggests "how much is shaded in" and Shane suggests "what shapes the parts are." The teacher records these ideas on the board as a list of important ideas for naming a fraction, but reformulates her question by making a comparison between the reasoning behind 1/3 and the reasoning behind 1/4. Daniel, who wrote 1/4 both in the last week's homework and during the lesson, takes the floor to explain.

> Well, what they're doing is one-fourth, they're saying that the rectang-the big rectangle, which is half of the shape, it can be cut into resemble two-two squares. But with a third, they're saying, they're treating the rectangle like one little square, so then they say that it's three. That it's one-third.

After reformulating the question, the teacher gets a rough idea from Daniel, which has potential to be a foundation for developing the complete mathematical explanation for the brown rectangle problem. The teacher first draws attention to the idea of filling in the line and then makes a comparison between two proposals (1/3 and 1/4) to elicit that making equal parts is an important idea for naming a fraction. Daniel's explanation is inaccurate (naming "resemble" rather than "same"), somewhat contradictory (he names the half of the whole as a big rectangle whereas the teacher names the whole as a big rectangle), vague ("treating the rectangle like one little square" to point out that they are not the same), and incomplete (missing the information why the right side of the big rectangle should be cut and addressing the production of adding the line rather than the reason of adding the line). However, his periphrastic statement adds details to Shane's idea and gives an access to the idea of equal. Instead of inspecting the details of Daniel's explanation, asking for repeating, or checking for agreement or disagreement, the teacher asks other students to scale up Daniel's explanation for developing key ideas of naming a fraction, but keeps eyes on figuring out differences between counting three parts and counting four parts. Amber explains "how the length is equal" at first but elaborates into "how all the parts are equal" with the teacher's request for repeating. Building on Amber's idea of "equal," the teacher summarizes that making equal parts is an important idea for naming a fraction and drawing a line creates an easy access to see the equal parts.

After modeling a complete mathematical explanation for the second problem, the teacher goes back to Alexa's proposal and asks students to compare reasoning between

1/2 and 1/4. The teacher draws a line around each of the two rectangles that Alexa identified and then asks students about the differences between naming it as 1/2 and as 1/4. Elis explains:



> That person was like, he or she probably doesn't understand that… they probably don't, they don't understand they could just make a rectangle, two rectangles and colored in half.

Instead of providing an answer for the teacher's question to make a comparison between 1/2 and 1/4, Elis provides his own explanation as to why someone might call it as 1/2 instead of 1/4. Without further specification, either verbal elaboration or gestural indication, Elis' explanation can be interpreted in two different ways. On the one hand, the phrase "they could just make a rectangle" can be interpreted as taking the left side of the big rectangle as a whole (taking the different size of the whole) and the term "two rectangles" can be interpreted as the left-upper piece of the big rectangle and the left-bottom piece of the big rectangle. On the other hand, it also could be interpreted that the drawing could be considered as 1/2 while maintaining the same size of the whole but expanding the shaded part. Because 1/2 is twice the size as 1/4, it might be more reasonable for students to adjust the amount of the shaded part rather than changing the size of the whole.

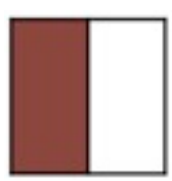
Clarifying the two rectangles (the left side of the big rectangle and the right side of the big rectangle) and reminding the students what Alexa ignored (the right side of the big rectangle), the teacher asks a more focused question: "what is it one-half *of*?" The teacher asks someone to go to the board and points to the rectangle that it is half of. Marcel goes to the board and points to the left side of the big rectangle. Getting a confirmation that the shaded area is 1/2 of the left side of the big rectangle, the teacher cuts off the other side of the big rectangle which Alexa ignored and then asks someone to explain why it is reasonable to call this 1/2.

Kurtis:	Using that each, there's one yellow part and one white part. What I think the person is trying to do is take away that line, separating white from yellow, and just make that one yellow, and so that one, put those together, that would be one-half.
Teacher:	If you put this (putting the right side of the big rectangle back) together? I think that's different. Can you come up and show what you're saying? 'Cause I think what Alexa was saying they ignore the other one, right? Alexa?
Alexa:	Huh?
Teacher:	You're saying the person ignores this one, right?
Alexa:	Yes.
Teacher:	So, can you show us what you're thinking about putting it back together, Kurtis?
Kurtis:	(comes to the board) They saw that there's a yellow [brown] one and white part. And there would probably take away the black [line] one, separate the yellow [brown] one from white. This makes that one whole yellow [brown] one and then put the blank [line] one back together that one is one-half.

In explaining why it is reasonable to call the second problem as 1/2, Kurtis fixes the size of the whole but adjusts the amount of shaded part. After Kurtis' explanation, several students express their agreements with Kurtis' explanation. Roddie explains:

> He means like, one you have yellow part up there, you make that whole one as just yellow, and put it back together, then it's one-half.

The teacher points out that Kurtis' idea is different from Alexa's idea, shades the left side of the big rectangle on the board, and asks again why the shaded brown is 1/2. Even though Kurtis changes the original problem, his effort to make sense of the incorrect answer contributes to the clarification about the whole.



Niena:	Because you shaded one-half of the rectangle.
Teacher:	And how did you know it's one-half? Why does one-half mean? Could we use reasoning about parts and equal?
Student:	(inaudible)
Teacher:	But, let Niena explain it. Why is that one-half?
Niena:	Because you shaded the… you… uhm, number three
Teacher:	How many parts are there?
Niena:	There's two parts.
Teacher:	Two parts and how many are-are they equal?
Niena:	Yes.
Teacher:	How many are shaded?
Niena:	One.
Teacher:	Okay, so the way Kurtis was seeing was this (tracing the big rectangle) is the whole, two parts, after you colored in this part and one of them is shaded, so that's one-half. Okay? Thank you, Kurtis.
Marcel:	It doesn't all equal one whole, but it's one-half shaded.
Teacher:	Sorry?
Marcel:	It doesn't all equal one whole, but it's one-half shaded.

Instead of giving turn to other students to simply repeat Niena's explanation, the teacher asks more focused questions, guided by the established knowledge about naming a fraction written on the board. After supporting Niena to complete her explanation, the teacher revoices how Kurtis saw 1/2 by binding the fragmented information together which has been built through several exchanges between the teacher and Niena, while explicitly pointing out the pieces in the drawing.

At the end, Marcel provides an insightful comment that Kurtis' drawing and Alexa's drawing are both 1/2, but they have different size of the whole. His comment is similar to what the teacher sets up for comparing between 1/3 and 1/4 and for comparing 1/2 and 1/4. This comment could be expanded to reveal another important idea about a fraction (i.e., a fraction is a relative amount of a part to a whole), but taking up this idea further sacrifices instructional time on reinforcing the concept of identifying the whole. After clarifying the referent of the whole in the proposal of 1/2, the teacher returns to Tatiana's proposal.

Teacher:	Tatiana said this (pointing to the left side of the big rectangle) and this (pointing to the right side of the big rectangle) as each one whole. The rest of you think that this whole thing (tracing to the boarder of the big rectangle) is the whole, and you are cutting it up.

	But, Tatiana noticed that here's a rectangle, just like the one you looked at a minute ago, here's one (making the border of the right side of the big rectangle as dark) and here's another one (making the border of the left side of the big rectangle as dark), so this part is the one (circling 1 in the 1 and 1/3). Is that right, Tatiana? That's the one. And then she looked over here and she called this one-third.
Student:	How does it one-third?
Teacher:	But what should Tatiana have called this because this isn't three equal parts, but it's what? How many equal parts are on this side?
Students:	Two.
Teacher:	Two. So, can you revise what you said? It wouldn't be one and one-third, one and what?
Tatiana:	Half.
Teacher:	One and a half. Good. So, what Tatiana is noticing is that you can call this one and half because you can have two wholes here, this is one of the wholes, and this is one of the wholes. I should write the word, whole. This is the other main idea that we need to have today is you have to think what is the whole.

Tatiana's initial explanation was mathematically and pedagogically challenging to decipher. Her initial explanation was linguistically vague, mathematically inaccurate, and logically contradictory by drawing two different referents of a whole when naming a fractional part and when naming a whole number part. The teacher, however, does not discard Tatiana's idea, but makes use of it to reinforce the key concept for naming a fraction—identifying the whole and making equal parts. Using the established knowledge, the teacher supports Tatiana to revise her initial proposal. The revised proposal of 1 and 1/2 has a different whole (half of the big rectangle) and names the unshaded parts instead of the shaded part, but it provides an opportunity to practice the established knowledge for naming a fraction. After revising Tatiana's idea, the teacher completes the list of ideas that need to be considered for naming a fraction on the board and addresses the importance of clarifying what they are calling the whole in naming a fraction. The key ideas that are listed on the board for naming a fraction are:

1. how many parts
2. equal
3. how much is shaded
4. what is the whole

The teacher then improvises another task to do some practices with naming a fraction with a different whole and wraps a whole-group discussion by reminding that it is important to clearly say what the whole is.

Summary

The EML 2007 students propose one answer (1/3) for the first part of the brown rectangle problem and discusses four proposals (1/4, 1/3, 1/2, and 1 and 1/3) for the second part of the brown rectangle problem (three are proposed by the students and one is nominated by the teacher). In explaining the brown rectangle problem on Day 6, the EML 2007 students initially miss the key idea of "equal" for naming a fraction for the first part of the brown rectangle problem; use the inaccurate language (e.g., "not even," "in half," and "a regular rectangle") which its intended meaning is different from the accepted mathematical definition; use demonstrative pronouns (e.g., "this"); grant geometric names incorrectly, incoherently, and indistinguishably; and have lack of specificity beyond counting (e.g., "one" instead of "one whole"), but do not have much difficulties with explaining in a public space and with hearing others' explanations in general.

For the first part of the brown rectangle problem, Tatiana proposes the correct answer (1/3) but provides the incomplete explanation (i.e., missing "equal"). The early tacit agreement on Tatiana's answer about the first part of the brown rectangle problem does not create a rich opportunity to develop the key idea for explaining the brown rectangle problem. For the second part of the brown rectangle problem, Lila first explains the correct answer (1/4) and addresses the key idea of "equal" by refuting another proposal (1/3). The key idea of "equal" emerges at the beginning of the whole-group discussion but it is not readily pick up by the teacher. Following that, the explanations for incorrect answers are provided. The initial proposed explanation for the correct answer (1/3) for the first problem by Tatiana ("three squares") is not identical to the initial proposed explanation for the incorrect answer (1/3) for the second problem by Stan ("three parts"). After hearing an explanation for each proposal, the EML 2007 students compare proposals (1/3 and 1/4 first and then 1/2 and 1/4 later). In comparing between 1/3 and 1/4, the proponents of 1/4 (Christopher and Marcel) defend their proposal but the proponents of 1/3 are not active in defending their proposal. The comparison between 1/3 and 1/4 leads to develop one of the key ideas for naming a fraction (making equal parts), whereas the comparison between 1/2 and 1/4 leads to develop another key idea for naming a fraction (identifying the whole). The EML 2007 students do not make a strong counterargument between 1/3 and 1/4 but more engage in discussing different interpretations of 1/2.

To support students' development of mathematical explanation for the brown rectangle problem, the teacher distributes an equal opportunity to explain rather than heavily controlling the sequence of proposals; seeks for different answers; addresses a contradictory role of agreement on the answer; legitimizes disagreement about the answer and places a mathematical value of disagreement; differentiates the use of "agree" or "disagree" by the correctness of the answer; makes an extensive use of the initial explanation rather than eliciting various versions of explanations and just leaving them unexamined; requests for repeating and revoicing the initial explanation and then checks back with the initial explainer; invites students to the board so that the students supplement their verbal explanation with pictorial representations; delays the evaluation of the answer and the rigorous inspection of the completeness of explanation at the beginning but addresses the incompleteness of explanation after developing the key ideas, and increases the level of mathematical supports over time (no substantive mathematical supports, including clarifying what the big rectangle refers to and remediating errors, during the set-up stage and the individual work).

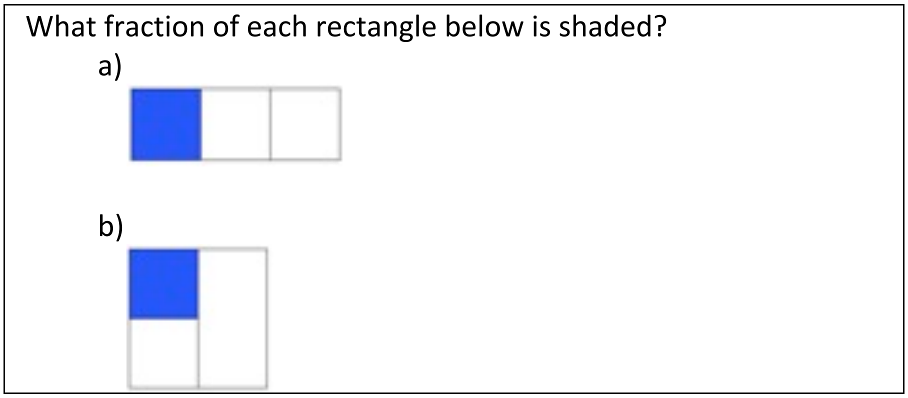
4.3. The Case of EML 2008

Preview

In the EML 2008, the brown rectangle problem is introduced on the second session of Day 1. For the first part of the problem (naming a fraction for the equally partitioned rectangle), only one answer (1/3) is proposed by Melody. After Melody's explanation, the teacher asks whether it is a complete explanation and whether everyone agrees with the answer. For the second problem (naming a fraction for the unequally partitioned rectangle), two proposals are made in a public space. The first proposal (1/3) is made by Chantal and the second proposal (1/4) is made by Kale. After hearing an explanation for each proposal carefully, the teacher does a quick survey of how many students think that the answer is 1/3 and how many students think that the answer is 1/4. The teacher contrasts two proposals and asks whether the answer should be 1/3 or 1/4. Karl first argues that the answer should be 1/3 because the problem comes out without a line. The teacher challenges Karl with Kale's idea about adding a line, but Karl maintains his position that the line should not be there. Next, Britney argues that the answer should be 1/4 because the fraction should be equal parts. Redirecting the attention to Karl's idea, the teacher asks whether a line is allowed to add. Unlike the EML 2007, the teacher explicitly addresses this mathematical issue. Calder agues that a line should not be added, Dalton argues that a line should be added, and Manoel argues that the line should not be added. Alexico argues that it does not matter whether a line is drawn or not. By engaging in the argument about whether or not the line should be drawn, the EML 2008 cohort collectively drafts a key idea of naming a fraction—making equal parts—which ultimately leads to the development of complete mathematical explanation for the brown rectangle problem. The extensive detailed analysis of the 20-minutes instructional interactions for teaching the brown rectangle problem in the EML 2008 is provided below.

Extensive Detailed Analysis

The second session of Day 1 in the EML 2008 begins with the brown rectangle problem. Unlike the EML 2007, the teacher posts two problems (the equally partitioned rectangle and the unequally partitioned rectangle) at the same time under one written problem statement.



After posting the brown rectangle problem[17] on the board, the teacher asks someone to read the problem statement written on the poster to make sure that everyone understands what the problem is asking. Saniya reads aloud the problem statement but replaces the word "each" rectangle to "the" rectangle. Noticing such a subtle difference, the teacher clarifies that the problem is asking students to write down a fraction for each rectangle, rectangle a) and rectangle b). Such a clarification would prevent students from misinterpreting the problem as asking them to provide one fraction that applies both for the equally partitioned rectangle and the unequally partitioned rectangle or from misinterpreting the problem as asking them to provide a fraction that applies only for the equally partitioned rectangle problem (because it is a non-square rectangle). Due to the obvious similarities between rectangle a) and rectangle b)—there are three parts and the size of the shaded area is the same—, lack of such a clarification about the problem statement might not provide a sufficient opportunity for students to see the important but latent mathematical idea embedded in the brown rectangle problem: (1) rectangle a) is equally divided into three parts but rectangle b) is not equally divided into three parts; and (2) a fraction is determined by comparing the relative size of a shaded part to the whole rather than measuring the size of a shaded part only. However, the teacher does

[17] The rectangle is shaded blue only in the EML 2008. For the convenience, I name the brown rectangle problem throughout the dissertation.

not clarify what "rectangle" refers to at this point. Tracing the boundary of each rectangle increases an accurate interpretation about the brown rectangle problem, but eliminates an incorrect answer that would introduce a key idea for developing a mathematical explanation—identifying the whole—at the early stage.

After a brief introduction about how to set up a notebook, the teacher gives time for students to work on the brown rectangle problem individually. During an individual work, the teacher mainly checks whether students write down a fraction for each problem but does not provide substantive mathematical support spontaneously unless students request for help. While the teacher circulates the classroom, several students seek help for clarifying the following mathematical issues:

- Reason for naming the diagram as a rectangle instead of a square: Linda
- The referent of "the rectangle": Ander
- Allowance, concerns, and questions around drawing a line: Adele, Linda, and Britney

The requests for clarifying the last two mathematical issues impact answers that students might bring in to the public space at a later point. Instead of addressing those mathematical issues publicly, the teacher responds to an individual student's request in a private space.

After three-minutes of individual work, the teacher convenes the class to launch a whole-group discussion and then shares her observation that most students get the same answer for the first problem but have different answers for the second problem. Most of the students, except Jacey, Melinda, and Linda, neatly wrote only one correct answer of 1/3 for the first problem in their notebooks. Jacey overlapped 1/3 with 1/2 (Figure 4.3) and Melinda crossed out something completely then wrote 1/3 next to her initial answer (Figure 4.4). Based on these records in the notebooks, it is not easy to track when they changed their answers and why. Unlike other students who wrote only one correct answer, Linda recorded multiple correct answers—equivalent fractions of 1/3—by adding additional lines in the drawing (Figure 4.5).

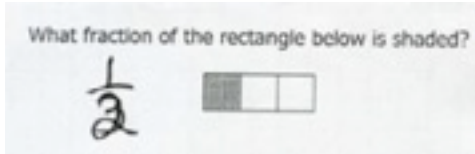


Figure 4.3. Jacey's written record for the first part of the brown rectangle problem

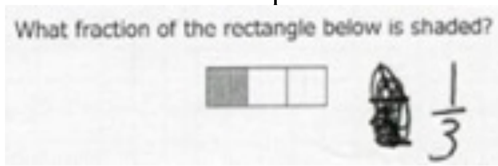


Figure 4.4. Melinda's written record for the first part of the brown rectangle problem

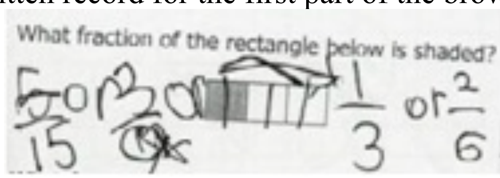


Figure 4.5. Linda's written record for the first part of the brown rectangle problem

Opening up a whole-group discussion, the teacher looks for someone who has not had a chance to talk in a public space yet. In some sense, selecting a student who has a particular mathematical idea enables a teacher to have a control of sequencing proposals. Given that most students came up with the same answer for the first problem, this method of giving the floor does not quite impact on sequencing, but reveals that the teacher has made an effort to distribute an opportunity to explain equally to all students rather than heavily controlling the sequence of proposals. Melody gets a turn and then explains:

> The answer to number one is one-third because there's three parts and only one is shaded in.

The teacher invites Melody to the board to write the answer, but Melody declines the invitation. Beyond often-discussed issues around authority, responsibility, and agency, the invitation to the board plays an important role in the development of mathematical explanation for the brown rectangle problem because the physical relocation creates opportunities (1) to repeat an initial explanation naturally; (2) to provide a visual supplement to a verbal explanation by pointing out pieces explicitly, especially in the case which a verbal explanation is heavily loaded with demonstrative pronouns (e.g., this); (3) to catch any comments or questions from audience with a bird's eye view; and

(4) to bounce off a mathematical issue between a proposer and audience. Because the brown rectangle problem was introduced on the first day of EML 2008, however, going to the board would not be an easy task for the students, including Melody.

Confronted with Melody's refusal to relocate herself in a public space, the teacher serves as a delegate to write the answer on the board but provides a support for Melody to continue her talk. In responding to the teacher's request about information on "how to write," Melody mechanically provides a direction, "One, you would write one and then a line under one and put three." After writing Melody's answer (1/3) on the board, the teacher asks Melody to repeat her initial explanation, while asking other students to make sure whether it is a complete explanation. Upon the teacher's request, Melody repeats:

> The answer is one-third 'cause there's three parts and that's the denominator and then the numerator is one 'cause there's only one part is shaded in out of the three.

In the repeated explanation, Melody uses the same term (i.e., three parts) but adds the information about what is corresponded to the denominator (three parts) and what is corresponded to the numerator (one part shaded). Melody adds a little bit of details, but still misses the key mathematical idea (i.e., "equal" parts) to be counted as a complete mathematical explanation for the brown rectangle problem. Unlike the initial explanation provided by Tatiana in the EML2007, Melody chooses "three parts" rather than "three squares" in providing her explanation for the first problem. The teacher revoices Melody's explanation and then asks other students whether it is a complete explanation and whether everyone agrees with 1/3. Unlike the EML 2007, the teacher explicitly draws students' attention to the completeness of the explanation and explicitly asks for an agreement with the answer for the first problem. As the EML 2008 cohort accepts the completeness of Melody's explanation and reaches an agreement on Melody's answer, the teacher moves on to the second problem. The teacher briefly checks with Delilah whether she has a different answer, but she does not offer any other answer. The teacher then makes a transition to the second problem.

For the second problem, three different answers are recorded in the students' notebooks: 1/4, 1/3, and 1/2. Because several students crossed out their initial answers completely and then added 1/4, the records themselves do not reveal the exact proportion of the students who produced the correct answer to the students who produced the incorrect answers during individual work. However, nine students (Alexico, Adele, Dalton, Galvin, Jack, Kale, Linda, Melody, and Melika) only recorded 1/4, while the remaining 17 students kept their incorrect answers, crossed out their original answers completely and then wrote the correct answer, or wrote both correct and incorrect answer in their notebooks. The teacher reminds the students that they produced different answers for the second problem and then gives Chantal the floor to explain first.

> Uhmm… I think it's one-third because there's three parts and one is shaded, but first when I first saw it I thought it was four, one-fourth.

Chantal provides a quite clear explanation at first, but creates complicated work for the teacher to deal with by laying out two mutually exclusive proposals at the same time. Chantal ends up with the incorrect answer of 1/3, but considered the correct answer of 1/4 at the beginning. This is an interesting case in that many students often change their mind from 1/3 to 1/4 once they begin to notice four parts instead of three parts, but Chantal once recognized four parts but might not be convinced by herself in staying with 1/4. At this point, eliciting an explanation of 1/4 from Chantal, analyzing how her explanation for 1/4 is different from her explanation for 1/3, and investigating further what made Chantal change her mind from 1/4 to 1/3 might be one route that a teacher could take, but it would impose a heavy burden on Chantal and limit the production and the use of a collective resource in developing a mathematical explanation by exclusively relying on one student's explanation for both competing proposals.

The teacher asks Chantal to repeat her explanation for 1/3 and Chantal repeats "Because there's one shaded and there are three parts." Chantal's explanation of her incorrect answer of 1/3 for the second problem ("There are three parts and one is shaded" in her initial explanation and "There's one shaded and there are three parts" in her repeated explanation) is exactly same as Melody's explanation of her correct answer of 1/3 for the first problem ("There's three parts and only one is shaded in"). After repeating Chantal's explanation, the teacher asks Chantal whether the reasoning for the second problem is the same as the reasoning for the first problem. Then she asks other students whether Chantal needs to add anything to her explanation. Unlike the discourse move that the teacher made after Melody's explanation of the correct answer for the first problem—whether everybody agrees with Melody—she does not ask whether the students agree with Chantal, but asks whether everybody understands her explanation. As no further comments are made, the teacher calls for a different proposal for the second problem by reminding her observation that half of students have one answer but half of students have another answer for the second problem. Once again, the teacher does not heavily control the sequence of proposals but asks someone to volunteer who did not have a chance to talk yet. Kale gets a floor to explain his answer of 1/4.

Kale:	I think it's one-fourth.
Teacher:	You think it's one-fourth? Why do you think it's one-fourth?
Kale:	Because, uhm…
Teacher:	(writes 1/4 on the board)
Kale:	cut a line, uhm… through, cut a line through, uhm…
Teacher:	You wanna come do it?
Kale:	No.
Teacher:	Okay, tell me what to do.
Kale:	Cut a line through the long part.
Teacher:	(Pointing to the right side of rectangle) Here?
Kale:	Yeah.
Teacher:	So what does that do?
Kale:	Makes, uhm, four blocks.
Teacher:	Okay. And?
Kale:	And one is shaded
Teacher:	Okay. So Kale says if you put the line through here, then you have four parts and one is shaded, so it's one-fourth.

The teacher invites Kale to the board, but, like Melody, he declines the invitation. Instead of passing the floor to another student, the teacher serves as a delegate to fully address Kale's idea in a public space. The turn-taking structure is one of the widely used tools to analyze the quality of classroom talk, but exclusively relying on it might miss an important aspect of the work to support students' development of mathematical explanation. In tracking Kale's explanation, it is revealed that the teacher provides sustained support for him to add details about the result of his action (i.e., making four blocks) beyond laying out the action he took to get the answer (i.e., drawing a line), but does not impose on making explicit why he adds the line at this point. After Kale sketches out where 1/4 comes from, the teacher revoices his explanation in a full sentence to bind the fragmented information together which has been built through several exchanges between the teacher and Kale. In revoicing Kale's explanation, the teacher also adopts the term "parts" which is aired by Melody and Chantal, rather than the term "blocks" or the term "squares."

The teacher then takes a quick survey of how many students got 1/3 and how many students got 1/4 by asking them to raise their hands, but provides a comment that it is not a vote. It indicates that the proposal supported by a majority of students is not counted as a base of mathematical decision-making. Rather, it is useful resource for a teacher to have a better sense of the composition of students' mathematical ideas, which might translate into customizing her prompts later. After a quick survey about these two proposals, the teacher continues to seek a different answer. Beyond these two proposed answers (1/3 and 1/4), another answer (1/2) is observed in the student's notebook. Because several students completely crossed out their initial answers and wrote the correct answer of 1/4, it is not clear how many students came up with 1/2, except Jacey and Jamila. Jacey overlapped 1/2 and 1/3 in her notebook (see Figure 4.6), so it would not be easy to notice while circulating the classroom.

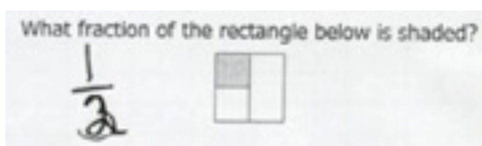


Figure 4.6. Jacey's answer for the second problem in her notebook

Unlike other students, Jamila provides very detailed explanation about her answer of 1/2 (see Figure 4.7). Her written explanation provides a clue that the geometric name of "rectangle" in the problem statement rather than "square" made her choose a different whole—a half of the whole—which is a non-square rectangle. Lack of understanding about the inclusive relationship between quadrilaterals (i.e., a square is a special kind of a rectangle) leads a student to take a different whole than being intended in the problem statement, but such a vagueness embedded in a problem statement creates an opportunity to develop a key idea for a complete mathematical explanation of the brown rectangle problem—identifying the whole. Jamila wrote a very detailed explanation in her notebook, but did not share her proposal in a public space. One reason might be that

Jamila began to recognize what is meant by a rectangle in the brown rectangle problem while hearing explanations of other proposals.

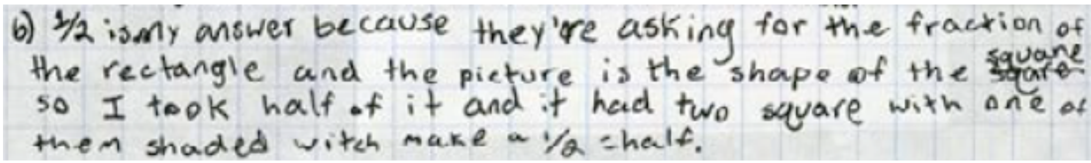

b) 1/2 is my answer because they're asking for the fraction of the rectangle and the picture is the shape of the square so I took half of it and it had two square with one of them shaded witch make a 1/2 = half.

Figure 4.7. Jamila's written record for the second part of the brown rectangle problem

In the EML 2007, the teacher listed all of the proposals that the students produced in their notebooks to the public space. On the other hand, in the EML 2008, she does not force students to nominate 1/2 further but mainly confronts two competing proposals (1/3 and 1/4) to develop a mathematical explanation. The teacher says:

> Okay, so can anybody, does anybody think they have a reason why it *should* be one of these or the other? 'Cause you're trying to *convince* the other people in the class. Obviously, not everybody in the class agrees here. So what could you use to *convince* people one of these answers or the other one?

It is noticeable how the teacher customizes her prompt according to the students' mathematical stances. In the context where four proposals (1/4, 1/3, 1/2, and 1 and 1/3) were made in the public space in the EML 2007, the teacher asked about the differences between the reasoning behind 1/3 and the reasoning behind 1/4 from a more neutral stance. On the other hand, in the context where only two proposals are made in a public space and the incorrect answer is more pervasive than the correct answer, the teacher uses the modal verb "should" which positions one's opinion much stronger than "could" or "might." It sets up a context in which a student needs to take a stronger stance about his or her idea and allows the teacher to allude that these two proposals—1/3 and 1/4—are mutually incompatible for naming a fraction in the diagram.

The choice of "convince" rather than "persuade" conveys the idea that students need to build a logical argument with the supportive data, to make someone believe one's argument, to change someone's perspective to agree with one's argument, and to leave little space for disagreement with one's argument. Reaching an agreement about the answer is a final mathematical goal to achieve, but the existence of disagreement provides a strong motive for elaborating, reinforcing, and defending one's explanation. In responding to the teacher's request, Karl explains:

> I think that it *should* be one-third. Because there is just, uhm… there's three, one, they cut in half, one, two, and then the whole one is three. So I think it *should* be one-third because that's the way it looks.

Compared to the previous statements made by Chantal ("I think it's one-third") and Kale ("I think that it's one-fourth"), Karl exhibits a stronger voice by using the modal verb of "should" which is largely influenced by the teacher's framing of the question. It is interesting to see that Karl uses the language that is often used by the students who explain the correct answer of 1/4 in adding a line (i.e., "cut in half"). One possible reason might be that Karl has been influenced by Kale's explanation when he added the line in the right side of the rectangle. It is not clear whether Karl adopts Kale's language, but his language choice arouses mathematical interest in why "cut in half" is counted only in the left side of the rectangle but is not further considered in the right side of the rectangle. Compared to the previous explanation provided by Chantal, Karl adds a ground to support his claim, but it is not a mathematically warranted reason (i.e., "the way it looks"). In addition, he just counts (i.e., one, two, three) but does not add references for counting (i.e., three parts). The teacher counts each of three parts with her finger, repeats Karl's explanation and then challenges Karl with Kale's idea about drawing a line. Karl defends his position:

> The line-it- it wasn't made to be there. So it-the way the problem came out was one-third.

In defending his position, Karl does not add a mathematically warranted reason but maintains his initial perception that the original problem comes out without a line, so the line should not be added. Once again, the teacher does not ask "Do you agree or disagree?" with Karl, but asks whether everyone understands Karl's explanation and then asks someone to explain the answer of 1/4.

Britney, who initially wrote 1/3 in her notebook, now defends her answer of 1/4. At the very beginning of the lesson, Britney exchanged an interesting conversation with the teacher in terms of drawing a line during individual work:

Britney:	So me and her have different answers, ‘cause she drew a line here-
Teacher:	You can do that.
Britney:	And she said one-fourth.
Teacher:	Yeah, you can do that if you want to.
Britney:	I got one-third, is that still […]
Teacher:	Well, does-you mean if you make a line, it changes the number?
Britney:	Yeah.
Teacher:	Okay. Then I guess you shouldn’t put a line, if you think it changes the number. Or you could make a line and say “With the line, it’s this number and without the line, it’s that number.”
Britney:	Huh?
Teacher:	You think it makes a different number, if you make a line?
Britney:	Yes.
Teacher:	So then you should write that in your notebook. You could say if I draw a line, then it’s this.”
Britney:	Okay.
Teacher:	That would be a very complete answer then, ‘cause you would be saying both the things.

During individual work, Britney noticed that her partner drew a line and had a different answer from hers. Even though she crossed out her initial answers in her notebook, the record indicates that Britney had two mutually exclusive answers for the second problem in mind. Britney recorded:

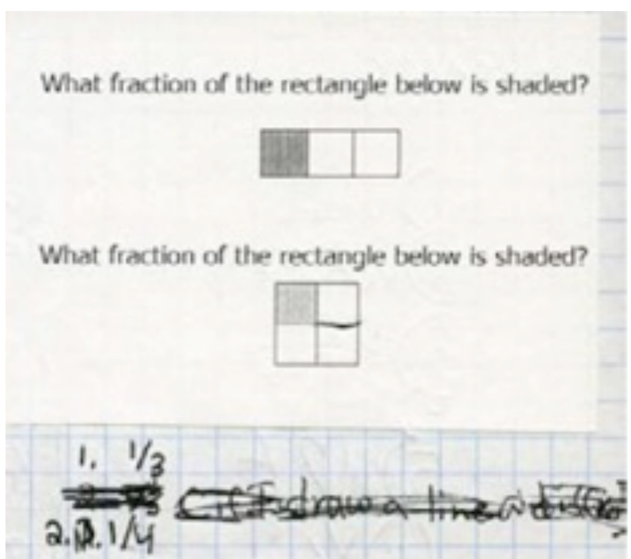


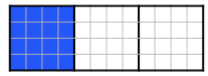
Figure 4.8. Britney’s written record for the brown rectangle problem

Along with the answers of 1/3 and 1/4 for the second problem, Britney added an explanation that drawing a line makes a different number even though she crossed out the explanation later. In some sense, Britney's initial idea is similar to students who argue that the line should not be added because it changes the problem. During a whole-group discussion, Britney defends her answer of 1/4, instead of 1/3. She explains:

> Britney: Well, because a fraction is something that's divided into equal parts and if it was-if the line wasn't there, it would-it wouldn't be equal parts, so…
> Teacher: Okay, we just added a word that we didn't have before. Britney, what was the word you just added that nobody has said yet?
> Britney: Equal.
> Teacher: Equal parts. We've been talking about the number of parts, like we said three parts (pointing to the first problem), and one, two, three parts (pointing to the second problem), but Britney just added a word, she said equal parts. And that's a very important word. What does equal mean in the case of these drawings? What makes- what makes something equal parts here? Why-there-why is this one (pointing to the first problem) equal parts?

Karl supports his claim with his generic perception (i.e., adding a line changes the problem), but Britney supports her claim with the mathematical concept (i.e., fraction is something divided into equal parts) and adds the reason for adding the line. In addition, Britney offers an important word that the teacher strives for in developing a complete mathematical explanation during a whole-group discussion—equal parts. The teacher provides comments that the first problem and the second problem both have three parts, but alludes the idea that "equal" makes a difference between two problems.

To reinforce the meaning of equal parts, the teacher goes back to the first problem and asks what makes the parts equal. Melody, who provided an initial explanation of 1/3 for the first problem, elaborates "Because they're divided into three squares that are the same amount of space." Britney's idea supports Melody to elaborate her initial explanation that was provided at the beginning of a whole-group discussion. It is interesting to notice that Melody chose the term "three parts" for the first problem at the beginning of discussion, but changes it to "three squares" while adding the information about the same amount of space. Building on Britney's idea about "equal parts," Melody elaborates it as the same amount of space and the teacher adds that it is also called the same area. The teacher utilizes the grids on the poster and suggests to count how many little squares are inside of each square. After proving that each square has 16 little squares inside, the teacher comments that the same amount of space is also called the same area.



The teacher asks whether Britney's idea ("equal parts") and Melody's idea ("the same amount of spaces") are helpful in figuring out whether the answer for the second problem should be 1/3 or 1/4. Again, the teacher frames the question using the modal verb of "should." Alexico defends his answer of 1/4:

Alexico:	'Cause if you split- like- what you did, if you split the, uhm, portion of the rectangle that was two squares
Teacher:	This one? (pointing to the right side of rectangle)
Alexico:	Yeah. Then it would make uhm, it would all be equal to one-fourth.
Teacher:	So if you split, Alexico is saying if you split this here (pointing to the right side of rectangle), then these four all have the equal area. What happens if we don't put the line in? 'Cause I think Karl said pretty clearly that the problem did not come out that way. So I think one question is, are we <u>*allowed to*</u> put that line in? Or do we <u>*need to*</u> put the line in? What do people think? <u>*Should*</u> we put that line in the drawing? Or <u>*should*</u> we <u>*not*</u> draw that line in?

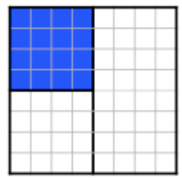
Alexico's language choice of "split the portion of rectangle" is mathematically quite similar to Kale's language choice of "cut a line through the long part." However, Alexico uses precise geometric names (i.e., squares) rather than generic names (i.e., blocks) and adds the key mathematical idea of "equal" in his explanation. Building a correspondence between verbal explanation and pictorial representation by pointing out the piece that Alexico refers to on the board, the teacher rechecks the reference of "the portion of the rectangle" with the author (Alexico) and make it clear to the audience (other students).

The teacher revoices Alexico's explanation, but does not close up the discussion yet. After hearing the rationale that the answer should be 1/4 two consecutive times, from Britney and Alexico, the teacher returns to Karl's claim and further challenges whether the line is allowed, needed, or should be drawn or not. In the context where the voice of supporting the proposal of 1/3 is quite strong, one of core tasks would be to provide a sufficient opportunity to defend the proposal of 1/3 and to check whether or not the students with the opposite opinion are fully convinced. On the surface level, the mathematical focus seems to be changed from "whether the answer should be 1/3 or 1/4" to "whether we should draw the line or not" but reformulating the question actually reinforces the key concept while dealing with very important mathematical issue in geometry. Calder defends his position that the line should not be drawn and Dalton defends his position that the line should be drawn.

Calder:	I don't think you *should* draw the line.
Teacher:	You don't think so? Or you do think so?
Calder:	I don't think so.
Teacher:	Why not?
Calder:	Well, because the fraction didn't come with the line, so I don't, if-
Teacher:	Speak up a little bit, please.
Calder:	Well, the fraction didn't come up with the line, so I-they-if it didn't come up with the line, I don't think the line *should* be there.
Teacher:	Okay, so Calder was saying the frac-the picture didn't come up with the line in it, so if the line wasn't there we *shouldn't* be putting it in. Okay, that's a perfectly reasonable thing to say the picture came a certain way. Who thinks we *should* put the line in and has a reason? 'Cause that's-that's pretty reasonable what Calder was saying. Does anyone have a reason why we *should* put the line in? Dalton, what do you think?
Dalton:	I think the line *should* be put in. Because if it wasn't put in, all the spaces wouldn't be equal and some people will get mixed up with the problem, and they put one-third.
Teacher:	So, you think we have to put the line in 'cause otherwise we don't have equal spaces? Okay, Britney?
Britney:	It's not a fraction, if it doesn't have-
Teacher:	Speak up.
Britney:	It's not a fraction, if it doesn't have equal parts. So, I think they were trying to trick you by not putting the line there.

Both Calder and Dalton defend their positions with a strong voice. Calder's argument is grounded on the assumption that drawing a line changes a problem but he is neither able to make use of any specific mathematical ideas to support his claim nor able to refute the opposing proposal. On the other hand, Dalton defends his position with the key concept of fraction (equal spaces) and refutes the opposing proposal. In supporting Dalton, Britney consolidates the idea that a fraction should be divided into equal parts and then makes a comment that "not" having a line tricks people.

In summarizing the argument, the teacher reminds the students that something needs to be divided into equal parts to name as a fraction and then asks the reason for drawing the line. Taking up the idea that the teacher used to prove that each of the three pieces has equal spaces for the first problem, Linda explains that without adding a line, the piece on the right side of the rectangle has 32 little squares, whereas the other two pieces on the left side of the rectangle have 16 little squares in it.

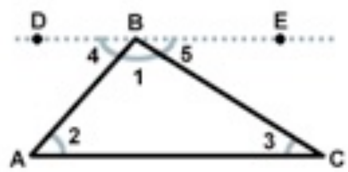


The teacher repeats Linda's explanation that they do not have three equal parts without a line because two pieces have 16 little squares but one piece has 32 little squares, while drawing attention from two students, Karl and Calder, who strongly positioned that the line should not be added. Manoel again defends his reason for not drawing a line:

Manoel:	Uhm, I don't think you ***should*** draw the line because uhm, uhm, if you… you can't change the problem. If you just draw the line, sure it'll be equal parts, but you'll be changing the problem.
Teacher:	So then what would you say for b, if someone says what part of this is shaded?
Manoel:	It ***should*** be one-third.
Teacher:	Okay, so then you're saying one-third would mean one out of three parts, but the parts aren't equal.
Manoel:	Exactly.
Teacher:	Okay.
Manoel:	'Cause if you draw the line, you're changing the problem and you're getting it wrong.

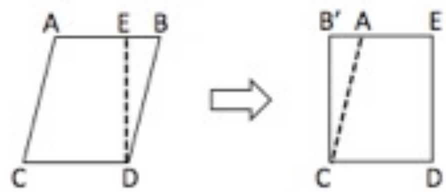
Taking up the mathematical issues that Calder, Dalton, and Britney brought up, Manoel strengthens his voice that the line should not be drawn. Manoel acknowledges that drawing a line makes equal parts, but he is more concerned that drawing a line changes the problem. His comment is mathematically interesting one because drawing a line—auxiliary lines—is one of the most important tools for proving in geometry. Figure 4.9 illustrates how auxiliary lines are used to prove geometric features, attributes, and theorems.

Example 1. Proving the sum of interior angles of a triangle is 180 by drawing an auxiliary line that is parallel to line AC.



- Drawing a line DE that is parallel to a line AC.
- Angle 4 is congruent to Angle 2 because alternate interior angles are congruent.
- Angle 5 is congruent to Angle 3 because alternative interior angles are congruent.
- The sum of angles 4, 1, 5 is 180 degrees.
- Thus, the sum of interior angels of a triangle is 180 degrees.

Example 2. Eliciting an area formula of parallelogram from an area formula of a rectangle by drawing an auxiliary line that is perpendicular to one of the bases.



- Drawing a perpendicular line of ED onto CD
- △DBE is congruent to △CAB′
- Thus, the area of parallelogram ABCD is same as the area of rectangle B′EDC: CD × ED

Figure 4.9. Examples of how the idea of drawing a line is used in geometry

In responding to the issue that Manoel raises, Alexico manipulates a key variable of the brown rectangle problem to support the claim that drawing a line does not matter.

Alexico:	I don't really-I don't even really think it matters if you draw the line or not. 'Cause if there are no lines, if there was a rectangle and the shaded rectangle or no other lines in the picture, then it would still be one-fourth.
Teacher:	Okay, so that's a new thing Alexico just said. Can you say that again and see what people think?
Alexico:	If there were no lines except for the rectangle, the-the-rectangle and the shaded piece of the rectangle, then it would still be one-fourth.
Teacher:	Why?
Alexico:	'Cause it-'Cause even without the lines, it's still one-fourth of the tri, of the rectangle. And it's the one-fourth

Teacher: Okay, so what Alexico is saying is that it doesn't really matter if you draw the line or not. It's still only one-fourth of this whole rectangle.

In the midst of the tension between two competitive claims—the claim that the line should be drawn and the claim that the line should not be drawn—, Alexico arbitrates the dispute by arguing that the line does not actually matter. In the EML 2007, Christopher and Marcel made the same claim that a line does not matter for naming a fraction to support the proposal of 1/4 for the second problem. The EML 2007 cohort did not establish the idea that drawing a line makes all parts equal at the point when Christopher and Marcel made the claim, so the teacher redirected students' attention to why it is important to fill in the line rather than further discussing whether drawing a line does not matter. In addition, the EML 2007 cohort did not make a strong opposing argument about why the line should not be drawn. On the other hand, the EML 2008 cohort already establishes the idea that drawing a line makes all parts equal and makes a strong argument that the line should not be drawn. The same mathematical claim is differently treated based on the established knowledge that a cohort collectively constructs and the mathematical stance that a cohort possesses.

To support the claim that a line does not matter, Alexico first introduces an example that has no lines in the rectangle without changing the cognitive demand of the task (still unequally partitioned rectangle). The teacher does not visually represent Alexico's first example in a public space, but asks Alexico to repeat his idea and then further investigates the reason. In responding to the teacher's further investigation, Alexico repeats his claim that it is 1/4 without any lines, but still leaves a gap between his conclusion (it is 1/4) and his claim (the line does not matter). Alexico's example is mathematically sophisticated one to challenge the idea that just counting the parts is not sufficient for naming a fraction because someone would call it 1/2 instead of 1/3 after deleting another existing line. However, it might not be convinced by some students who strongly believe that the line should neither be added nor be deleted from the original drawing. The example that Alexico verbally outlines looks like Figure 4.10.

Figure 4.10. Alexico's first example to support the claim that the line does not matter

As the teacher repeats Alexico's initial explanation, Alexico jumps in and suggests another example to strengthen his claim.

Alexico:	Also if the big long, the long part was shaded, that would make it one-half, even if there was a line going through the middle.
Teacher:	So you're saying if we had that same diagram
Alexico:	Exactly the opposite
Manoel:	You've convinced me.
Teacher:	And what would you say?
Alexico:	And the long part was shaded, then it would-the-the long, that problem would be one-half. It would still be one-half, the shaded part would be one-half.

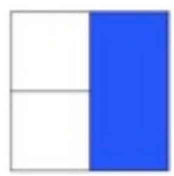


Figure 4.11. Alexico's second example to support the claim that the line does not matter

Alexico chooses the language of "the rectangle" to indicate different sizes of a rectangle (both for the whole and the shaded part) in the first example, but chooses the language of "the long part" to indicate the right side of the rectangle more clearly. To support his claim, Alexico devises another example by skillfully manipulating the key variable of the brown rectangle problem, without changing the cognitive demand of the task (still unequally partitioned rectangle). The original brown rectangle problem does not have a sufficient line, so it makes difficult to see four equal parts to name it as 1/4. On the other hand, Alexico's suggestion has an extra line which makes it difficult to see two equal parts to name it as 1/2. The two examples that Alexico introduce, which unequally partitioned the rectangle in the other ways, are effective to challenge the idea that just counting the parts is not sufficient to name a fraction, but the first example changes the line whereas the second examples does not change the line.

Even before Alexico fully provides details of his reasoning for the new example, Manoel who has a strong conviction that the line should not be added, immediately expresses that he is convinced by Alexico's idea. The teacher represents Alexico's second example on the board and then asks how many people think Alexico's idea is right and why it is right. Saniya and Chantal take the floor to explain.

Saniya:	Because they're both the same, but the other one just has a line through it.
Teacher:	Okay, so you're saying this (covering the right side of the rectangle) is the same as what? As what, Saniya?
Saniya:	As the other half.
Teacher:	So that one is one-half, but there are one, two, three parts, and only one is shaded. Why are you calling that one-half? Chantal?
Chantal:	Because… it's… uhm.. 'cause… I don't know.
Teacher	Go ahead. It's what?
Chantal:	Has- the same length, but if the other one has the line through it.
Teacher:	Okay, it's the same length, but the other one has the line through it. We actually could say it takes up the same space, like what Melody said.

In explaining why it is 1/2, both Saniya and Chantal attend to the idea of "same" rather than just counting the number of pieces. It is noticeable that the teacher gives a turn back to Chantal, who initially proposed 1/3 for the second problem and named the fraction based on the number of parts, to explain why it is called 1/2 despite of the three parts in Alexico's second example. The teacher summarizes a discussion by emphasizing that it is important to make equal parts to name a fraction rather than just counting all the different sizes of parts and wraps up the discussion by asking students whether they would like to keep their original answer or change their minds and write down a reason for their change in their notebooks.

Summary

The EML 2008 students propose one answer (1/3) for the first part of the brown rectangle problem and propose two answers (1/3 and 1/4) for the second part of the brown rectangle problem. In explaining the brown rectangle problem on Day 1, the EML 2008 students initially miss the key idea of "equal" for naming a fraction; decline the invitation to the board for explaining (e.g., Melody, Kale), use the pre-defined mathematical term without specifying the underlying concept behind the term (e.g., denominator; numerator); use the geometric-feature-free terms (e.g., "three parts," "four blocks," "three" rather than "three squares"); use the inaccurate language (e.g., "in half") which its intended meaning is different from the accepted mathematical definition; and describe an action taken (e.g., "cut the line through") rather the reasoning (e.g., why it is important to draw a line?), but actively participate in defending their own proposal and convincing others.

For the first part of the brown rectangle problem, Melody proposes the correct answer (1/3) but provides the incomplete explanation (i.e., missing "equal"). The early agreement on Melody's answer about the first part of the brown rectangle problem does not provide a rich opportunity to develop the key idea for explaining the brown rectangle problem. For the second part of the brown rectangle problem, Chantal first explains the incorrect answer (1/3) and then Kale explains the correct answer (1/4). The initial proposed explanation of the correct answer (1/3) for the first problem by Melody ("three parts") is same as the initial proposed explanation of the incorrect answer (1/3) for the second problem by Chantal ("three parts"). After hearing the explanations of two competing proposals, the EML 2008 students build arguments to convince why the answer should be 1/3 or 1/4. Both the proponents of 1/3 and the proponents of 1/4 are very active to defend their proposals. In the process of convincing each other, Britney brings up the idea about the role of drawing a line ("making equal parts"), whereas Karl, Calder, and Manoel argues that drawing a line changes the problem. The high tension between two competing proposals is resolved by Alexico's two improvised examples. The EML 2008 students alter the original mathematical task to support their claim but do not change the cognitive demands of the mathematical task. Rather, the EML 2008 students skillfully manipulate the key components of the mathematical task to convince others (e.g., 1/4 which is unequally partitioned into two parts; 1/2 which is unequally partitioned into three parts).

To support students' development of mathematical explanation for the brown rectangle problem, the teacher distributes an equal opportunity to explain rather than heavily controlling the sequence of proposals; seeks for different answers; uses sufficient waiting time and serves as a delegate to make the idea to be fully addressed in a public space; legitimizes disagreement about the answer; makes an extensive use of the initial explanation rather than eliciting various versions of explanation and then just leaving them unexamined; requests for repeating and revoicing the initial explanation; gives a turn back to the students who initially proposed the incomplete explanation; draws an attention from the students who initially proposed the incorrect answer and continuously involve them in constructing an explanation; invites students to the board so that the students supplement their verbal explanation with pictorial representation; delays the evaluation of the answer and the rigorous inspection of the completeness of explanation at the beginning; confronts competing ideas persistently until the students are fully convinced; uses the grid to prove the equal parts; and increases the level of mathematical supports over time (no substantive mathematical supports, including clarifying what the rectangle refers to and remediating errors, during the set-up stage and the individual work).

4.4. The Case of EML 2009

Preview

In the EML 2009, the brown rectangle problem is introduced on the first session of Day 4. After posting the first problem on the board, the teacher asks students to write down the fraction and the reasons in their notebooks individually and then asks them to talk with a partner to agree on the answer. For the first part of the problem (naming a fraction for the equally partitioned rectangle), the correct answer of 1/3 is first proposed by Aiyana and Amelia. The EML 2009 cohort reaches a collective agreement about the answer, but the teacher continues to elicit different numbers. Sandra proposes 2/3 by naming not shaded parts and Callie proposes 2/6 by adding additional lines. For each proposal, the teacher asks whether the students have any comments and checks whether they agree or disagree.

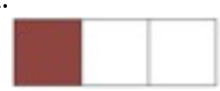
After eliciting an explanation for these three proposals for the first problem, the teacher posts the second problem on the board and asks students to write the fraction and the reasons in their notebooks, then asks them to talk with a partner to agree on an answer. Tiara and Jacqueline share their explanation for the proposal of 1/4. The teacher checks whether anyone has a question and makes sure everyone can hear the explanation. As no other comments are made, the teacher challenges the proposal of 1/4 by introducing the common incorrect answer of 1/3. Malik interprets that someone calls it 1/3 because there are three spaces and one shaded. The teacher asks whether it is reasonable to name it 1/3 and seeks for someone who agrees with the proposal of 1/3. However, no one agrees with the incorrect answer of 1/3. When the teacher asks for a reason for why it cannot be called 1/3, Teri offers the idea that it is not equal, but mainly addresses different shapes—a rectangle and two squares. With the teacher's continuous effort to elaborate the language of "not equal," Mannis reiterates "not equal" and Marlais elaborates that it is not the same size. After the teacher confirms that the three parts are equal for the first problem but the three parts are not equal for the second problem, she draws students' attention to why it is important to draw a line like Tiara and Jacqueline did and develops the idea that "making equal parts" is important in naming a fraction. The extensive detailed analysis of 24-minutes of instructional interactions for teaching the brown rectangle problem in the EML 2009 is provided below.

Extensive Detailed Analysis

After finishing a brief discussion about the warm-up problem, the teacher makes a transition to the brown rectangle problem on Day 4 in the EML 2009. Posting the first part of the brown rectangle problem (naming a fraction for the equally partitioned rectangle) on the board, the teacher asks students not to raise their hands or to say an answer but to write down the fraction in their notebooks. By preventing some knowledgeable students from blurting out an answer, the teacher ensures that all of the students have a private space to fully engage with the problem. The problem statement is written on the poster, but also verbally provided by the teacher with a slight variation: "what fraction of this shape is shaded in?" and "what fraction of that rectangle is shaded in or shaded brown?" As in the previous years, the teacher does not further clarify what "the rectangle" refers to at this point.

What fraction of the rectangle below is shaded in?

A.



A few seconds later the teacher asks students to write down the reason for the number they decided upon. Unlike the EML 2007 and the EML 2008, the teacher deliberately asks students to write down an explanation in their notebooks during individual work. Given that offering a verbal explanation in a public space is a daunting task for students, producing a written explanation in a private space beforehand supports students to reduce anxiety about a public speech, to build a logical structure of their explanation, to create more concise and less repetitive explanation, and to prepare a reference for their verbal explanation. A glimpse into students' written explanations would reveal whether or not the written explanation supports students in offering a verbal explanation, whether or not the written explanation is consistent with or transferable to verbal explanation, and to identify any inherent challenges in producing a particular form of explanation.

While the students are working on the problem individually, the teacher circulates the classroom to mainly check whether they write down answers and reasons in their notebooks, but does not provide any substantial mathematical supports during individual work. Unlike the EML 2008 cohort, none of the EML 2009 cohort members makes a request for the teacher to clarify the brown rectangle problem. After three-minute of individual work, the teacher asks students to talk with a partner to see if they agree on the number and how to explain it. In the EML 2007 and the EML 2008, the teacher made a direct transition from individual work to a whole-group discussion. On the other hand, in the EML 2008, the teacher asks students to talk with a partner to agree on the answer and to have them practice their explanations before launching a whole-group discussion.

All of the students record the correct answer for the first problem in their notebooks, but vary in the level of details in explaining what fraction of the rectangle is shaded in. Table 4.1 categorizes the students' written explanations by (1) the fractional components (i.e., shaded part, unshaded part, and the total number of parts in the whole) and (2) an indication about equal. Among 25 students, 17 students provide information about both the shaded part and the total number of parts in the whole. Two students provide information only about the shaded part and one student provides information only about shaded part and unshaded parts. Two students indicate about equal in explaining for the first problem, but Akilah does not mention the total number of parts in the whole and Nina does not provide any details about the shaded part and the total number of parts in the whole. As these records indicate, none of the students in the EML 2009 produce a complete mathematical explanation for the first part of the brown rectangle problem in their notebooks.

Table 4.1. Type of students' written explanations for the first part of the brown rectangle problem

The components of explanation	No Indication about Equal	Indication about Equal
Shaded parts	Evan	Akilah
Shaded parts Unshaded parts	Jacqueline	
The total number of parts in the whole		
Shaded parts The total number of parts in the whole	Aiyana, Alvan, Callie, Dante, Elina, Jana, Malik, Manley, Marlais, Marcellus*, Mannis, Natania, Riya, Ricky, Tiara*, Teri	
Shaded parts Unshaded parts The total number of parts in the whole	Tonya	
No details about shaded parts and the total number of parts in the whole		Nina
Other		
No explanation provided	Amelia, Collin, Sandra	
No records	Levi	

About a minute later, the teacher asks a pair to explain what they agreed on for the first part of the brown rectangle problem. Among four students (Aiyana, Callie, Elina, and Tonya), who raise their hands to explain, the teacher gives a turn to Aiyana and her partner, Amelia. A teacher could give a turn to a student based on completeness or incompleteness of written explanation in the notebook or could give a turn to a pair who both raise the hands. Elina and Tonya are the only pair who both raise their hands, but the teacher gives a turn to Aiyana and Amelia. In the lesson plan, the teacher specifies one of her goals as:

> To help each student learn to express and explain mathematics in class. (Try to make sure that these students get turns to talk or go to the board: Collin, Dante, Amelia, Jacqueline, Tiara, Teri, Alvan. Check in with and possibly add support for these students.) This list is based on things I noticed or that happened yesterday. In addition, I need to pay attention to Teri.

Giving a turn to Aiyana and Amelia allows the teacher to achieve her instructional goal to provide an opportunity for the targeted students to talk. The teacher first asks Aiyana and Amelia whether they agreed on. In the notebooks, Amelia does not provide her reasoning and Aiyana writes "It's 1/3 because there are 3 boxes and 1 shaded in." Aiyana, who wrote an explanation in her notebook, shares her explanation to the class:

> We both thought it was one-third, and that there were like three-like if-like anytime there's a fraction you try to count the boxes or circles, and see how many there are. Then-then you see how many are sh-then you see how many are shaded in. And, like however how many boxes or circles you have, you put it where the denominator is, and how many that are shaded in-you put in where the numerator is.

Aiyana first expresses an agreement on the answer with her partner, Amelia, and then provides a more detailed explanation than she actually wrote in her notebook. She first lays out the specific number (i.e., three), but soon expands her explanation to a general case (i.e., how many boxes or circles). Aiyana's verbal explanation is characterized as (1) providing more general idea about a fraction rather than being specific to the brown rectangle problem (i.e., "how many boxes" rather than "three boxes"); (2) making use of pre-defined mathematical terms (i.e., denominator and numerator); (3) mathematically incomplete (no indication about equal); (4) neither being specific to an explanation for the area model of fractions nor being transferable to the explanation for the number line model of fractions—her explanation is more of an explanation for the set model of fractions; and (5) repeating the same information three times.

Aiyana fluently provides an explanation, but talks too fast for other students to catch up. The teacher praises Aiyana's providing an explanation with lots of parts but asks either Amelia or Aiyana to repeat the explanation so that everyone can hear it. By doing so, the teacher might aim to create another opportunity for Amelia to provide an explanation in a public space. Aiyana again repeats her initial explanation:

> One-third because anytime there's a fraction you try to see how many-you try to see how many boxes or circles are in it, and then you put that where the de-denominator is. Then-then you go up to the numerator and you see how many are shaded in and- however-how many-many are shaded in, you put up where the numerator is.

Aiyana's repeated explanation is quite same as her initial explanation. She uses the pre-defined mathematical terms (i.e., denominator and numerator), but does not offer the key idea that underlies the concept of denominator which refers to the number of equal parts. The teacher asks whether someone has comments on Aiyana or Amelia's answer, but no one makes comments. The teacher then invites Natania to the board to write the answer. Natania writes 1/3 on the board and the teacher asks her to explain how she wrote it. Natania explains "This one is for one-third because there are three parts and one is shaded." Instead of repeating Aiyana's explanation, Natania provides her own explanation about the answer. Aiyana's explanation is more generic idea about a fraction, but Natania's explanation is more specific to the brown rectangle. In addition, Natania turns her attention to the part-whole relationship rather than describing it as separate entities of boxes or circles. Except the word-choice ("three squares" in her written explanation but "three parts" in her verbal explanation), Natania's verbal explanation is quite the same as her written explanation. The teacher does not make a comparison between Aiyana's explanation and Natania's explanation, but asks other students to comment, agree, and disagree.

Everybody agrees with the answer for the first part of the brown rectangle problem, but the teacher asks students whether they see another fraction in the diagram by saying that "I asked you how much of the rectangle is shaded in. But, do you see another fraction in that same rectangle that I didn't ask you about?" By leaving out the shaded color in the question, the teacher might open a possibility to come up with naming unshaded parts rather than shaded parts because unshaded parts could be interpreted as shaded white as well. The teacher continuously makes an effort to elicit another answer. All of the students record only 1/3 in their notebooks, but six students (Aiyana, Callie, Marlais, Marcellus, Malik, and Sandra) raise their hands to share their improvised proposals to the class.

The teacher gives a turn to Sandra. Sandra proposes 2/3 and then explains "Because there are two not shaded in and there are three boxes and only one is shaded in." Upon the teacher's request to say it louder, Sandra repeats "There are three boxes and only two are not shaded in. But there is one that is shaded in." Sandra wrote only 1/3 in her notebook, but improvises 2/3 by the teacher's attempt to elicit another answer.

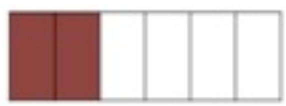
Even though Sandra names unshaded parts rather than shaded parts, her proposal has a potential contribution to develop the understanding that the sum of shaded part (1/3) and unshaded parts (2/3) equals to the whole (1).

The teacher praises Sandra's giving an explanation and checks whether others can see 2/3 that are not shaded. Evan explains that he can see 2/3 from the drawing and shows his agreement with what Sandra said. At this point, Callie expresses that she has another idea, but the teacher first invites someone to write the number that Sandra proposed to the board and uses this as an opportunity to repeat Sandra's explanation. Manley writes 2/3 on the board and explains:

Manley:	Like Sandra said, the two parts that are not shaded and the parts that is shaded-it'll be two-thirds because-like-it's like you are taking one away. And so it's-two-thirds, I mean two away.
Teacher:	So you're describing the brown parts as something that was taken away?
Manley:	(nodding his head)
Teacher:	So, what is the three referring to then?
Manley:	I mean like… the… well, let-let- okay. The brown part is like the-the part… I don't know how to explain it.
Teacher:	Well, what's the three referring to?
Manley:	There's three in all, there's three.
Teacher:	Can you show us the three in all in what you're looking at on the diagram? No, not in your writing, in the diagram.
Manley:	Oh.
Teacher:	Can you show in the diagram where the three is?
Manley:	One, two, three (points to each square).
Teacher:	Okay. And then the two is referring to what in the diagram?
Manley:	These two (points the two unshaded squares)
Teacher:	Do people agree with how he's explaining that?
Students:	Yes.

Unlike Natania who provides her own explanation about 1/3 after writing the answer that Aiyana proposed on the board, Manley begins his explanation by referring to Sandra. Aside from the word-choice ("parts" instead of "boxes"), Manley exactly captures how Sandra got 2/3 but tries to translate "not shaded" into a different concept. He first translates "shaded" as "taking away" but later changes "not shaded" as "taking away." The teacher captures his first translation and moves on linking the number to the drawing, but Manley tries to clarify what he means by "taking away." Getting confused by himself, Manley stutters and gives up on explaining further. It does not actually matter whether "shaded part" is "taking away" or "unshaded parts" are "taking away," so the teacher redirects Manley's attention to the link between the numerical representation he wrote and the pictorial representation and then asks him to point out the referent about "3" and the referent about "2" in the diagram. After Manley points to the pieces in the diagram, the teacher asks other students whether they agree with Manley's explanation.

After checking the agreement on Manley's explanation, the teacher moves onto Callie's idea. Callie only records 1/3 in her notebook, but improvises 2/6 by the teacher's attempt to elicit another answer. After looking at her notebook for a few seconds, Callie explains "I see… six…six squares and two shaded in." The teacher invites Callie to the board to show how she sees the fraction. The teacher checks whether other students understand what Callie said, but the students respond in a mixed way (half of them say "yes" and the other half say "no"). Coming to the board at the teacher's invitation, Callie draws three vertical lines on the drawing and then explains:



> I… first, I made three lines. I added two more lines and then I saw that two were shaded in. So that means that those are six rectangles and two are shaded in.

In proposing 2/6, Callie first names it as "six squares" in her seat but later corrects it as "six rectangles" after drawing lines on the board. Callie's idea—adding additional lines—is a key to get the correct answer for the second problem and to develop a complete explanation for the brown rectangle problem. The teacher does not further probe why she adds lines, whether it is okay to draw lines, and why it is important to draw lines at this point, but makes sure whether other students can hear Callie's reasoning. The teacher asks someone to repeat what Callie said, but both Marlais and Ricky say that they could not hear. The teacher reminds the norm of speaking louder and gives a turn to Nina. Nina explains:

> Uhm, she-she said that she added uhm, three more lines and got six and there was two shaded in. So that's three-I mean, two-sixths.

Nina repeats Callie's explanation, but leaves out the referent of six (i.e., "six" instead of "six rectangles"). The teacher checks back with Callie whether Nina captures her explanation and asks Ricky whether he could hear the explanation. The teacher then asks the students whether they agree on Callie's answer and gives a turn to Ricky. Ricky explains:

> Because if you add six- if you put those lines there, there's six and if you put it all together, there's only two are shaded in, which make it two-sixths.

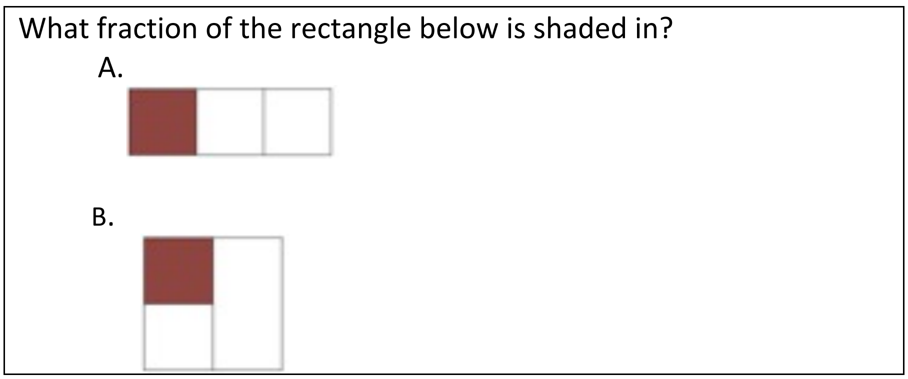
Like Nina, Ricky omits the referent of six (i.e., "six" instead of "six rectangles") but could repeat the explanation at this time. The teacher asks whether others can hear Ricky's explanation, but not every student could hear him. Reminding the norm of speaking loud enough, the teacher asks Ricky to explain again:

> Out of uhm, out of all of the-out of all of the ones, only two are shaded in. Out of the six, only two are shaded in, which would make it two sixths.

In the repeated explanation, Ricky misses the information about adding lines but captures other parts well. The teacher then asks the students whether anyone disagrees with Callie or has questions about it. As nobody makes comments, the teacher checks how to see 2/3 with Elina and how to see 2/6 with Nina. The EML 2007 cohort and the EML 2008 cohort spent about a minute discussing the first part of the brown rectangle problem, but the EML 2009 cohort spends about ten minutes discussing the first part of the brown rectangle problem. The EML 2009 cohort not only makes three proposals for the first part of the brown rectangle problem, but also does some practices with seeing and hearing others' proposals and are invited to make comments, agree, and disagree after each proposal is made. In the EML 2009, seeking for agreement or disagreement serves as accepting the answer or repeating the explanation, rather than challenging the proposed answer.

Sandra's proposals contributes to developing the idea that the sum of the shaded part and unshaded parts equals to the whole, whereas Callie's idea contributes to the idea that adding additional lines, under the condition that equal parts are preserved, produces equivalent fractions. However, these two proposals do not make a significant contribution to the development of complete mathematical explanation for the brown rectangle problem. Unlike the second part of the brown rectangle problem, the absence of incorrect answers, by the mathematical design of the first part of the brown rectangle problem, does not create an opportunity to enrich the mathematical explanation.

After ten-minute of whole-group discussion about the first problem, the teacher then peels off the paper which covers the second problem on the poster and asks students to work on the problem with a partner, not alone, and then asks students to talk with a partner to agree on what fraction of the rectangle is shaded in.



All of the students, except Akilah and Tiara, have clear records of 1/4 in their notebooks. Akilah and Tiara crossed out something completely and rewrote 1/4, so it is difficult to see what they came up with before a whole-group discussion. Table 4.2 categorizes the students' written explanations by three components: (1) the fractional components (i.e., shaded part, unshaded part, and the total number of parts in the whole), (2) indication about adding a line, and (3) indication about equal parts. Despite all of the correct answers, none of the students produces a complete written explanation for the second part of the brown rectangle problem in their notebooks. More students (Mannis, Jana, Marlais, Marcellus, Natania, and Riya) indicate the idea of "equal" for the second problem than the first problem in their notebooks. Two students (Aiyana and Nina) who indicate "equal" for the first problem do not indicate "equal" for the second problem.

Table 4.2. Type of students' written explanation for the second part of the brown rectangle problem

The components of explanation	No Indication about Adding a Line		Indication about Adding a Line	
	No Indication about Equal	Indication about Equal	No Indication about Equal	Indication about Equal
Shaded parts	Evan	Mannis		
Shaded parts Unshaded parts				
The total number of parts in the whole	Tiara		Manley, Teri	
Shaded parts The total number of parts in the whole	Ricky		Aiyana, Dante	
Shaded parts Unshaded parts The total number of parts in the whole				
No details about shaded parts and the total number of parts in the whole			Alvan, Callie, Elina, Malik, Tonya	Jana, Marlais*, Marcellus*, Natania, Riya
Other	Sandra			
No explanation provided	Akilah, Amelia, Collin, Jacqueline, Nina			
No records	Levi			

After three-minute of partner work, the teacher convenes the class to launch a whole-group discussion about the second problem. Jacqueline and Tiara, who are both the targeted students for explaining in a public space, are invited to explain what fraction of the rectangle is shaded in for the second part of the brown rectangle problem. Jacqueline does not have a record of her reasoning and Tiara writes " I think that it is 1/4 because you can make another box and make it four boxes." When Tiara gets a turn to provide her explanation in a public space, she explains "We got one… one-half over four, one over four." The teacher asks Tiara to repeat, and Tiara adds more details in her explanation. Tiara repeats:

> Well, we got it because we-we put a line through this big box right here and one was shaded,-one was shaded in, and three wasn't, so we got that it was one-fourth.

Tiara's verbal explanation adds more details than she actually wrote in her notebook, but still misses the key idea of "equal" in her explanation. The teacher invites both Jacqueline and Tiara to come up to the board to show their work on the diagram, but asks them to speak a little bit louder on the board. Jacqueline draws a line on the right side of the big rectangle and Tiara explains " We-we put a line through this big box and we made it into four and one is shaded."

Tiara adds the information further, but her explanation is hard to hear. The teacher notices that other students could not hear Tiara's explanation well, so asks the pair to speak a little bit louder again. This time, Jacqueline takes a turn and explains:

> We put a line right here (pointing to the right side of the big rectangle), and we put a line that's one-fourth because it's shaded in right here.

Jacqueline repeats the information that they put a line on the right side of the big rectangle but does not provide details about the shaded part and the total number of parts in the whole. The teacher asks Jacqueline to write the answer on the board and checks whether anyone has comments or questions for them. As no one asks a question, the teacher asks her own question to the class.

> I have a question. I'm gonna put up another copy of this drawing. I used this exact same drawing last summer in the-in the lab class, and it's not the answer that the kids told me. So I wanna know what you think about this. What most of the kids in the lab class last summer said for this one is they said this was one-third, just like they said for this one (pointing to the first problem). In fact, I can put up that picture too. Can someone tell me, not if you agree or disagree, but why do you think they're calling it one-third? Can someone see what they-the kids in my class last year were talking about? Here's the other drawing. So, this is the A drawing (the equally partitioned rectangle) and this is the B drawing (the unequally partitioned rectangle), Okay? So most of my students last summer said that that was one-third. Can someone explain what they were thinking about? Yes, Malik?

The incorrect answer of 1/3 violates one of the key ideas for naming a fraction—making equal parts—thus it plays an important role for developing a complete mathematical explanation for the brown rectangle problem. The absence of this incorrect answer makes it difficult to explicitly address the taken-for-granted idea of "equal" in the proposed explanation even though all of the students produce the correct answers for the brown rectangle problem. Partly because the teacher might be aware that all of the students produced the correct answer of 1/4 for the second problem during partner work and partly because the class already spent a fair amount of instructional time on eliciting different answers for the first part of the brown rectangle problem in the EML 2009, the teacher promptly introduces the incorrect answer of 1/3 to students.

While introducing the incorrect answer of 1/3 to the students, the teacher engages in some interesting mathematical work. First, the introduction of the incorrect answer is accompanied by making a reference to the first problem. Because of the seemingly geometric similarities between the first problem and the second problem—both are composed of three parts and the absolute size of the shaded part is the same—, making a reference to the first problem enables the students to see what makes someone to see 1/3, not 1/4, and provides a quick way to compare the differences between the first problem and the second problem. The same mathematical work was done by the teacher in the EML 2008, when Chantal provided the same explanation of her incorrect answer of 1/3 for the second problem as Melody's explanation of her correct answer of 1/3 for the first problem. Second, to protect the mathematical idea to be discussed, the teacher asks the students neither to agree nor to disagree, but to try understanding the reasoning behind the answer.

As the reason for why someone might answer 1/3, Malik explains "They see three spaces and one shaded." With the teacher's request to repeat the explanation, Malik provides the same explanation "because they see three spaces and one shaded." The teacher counts the three pieces on the second problem and checks whether other students have the same thought. The teacher asks whether it is reasonable to call it 1/3 and seeks for someone who agree with it:

> Well, isn't that right because it's one out of three then. So wouldn't we call that one-third? (writing 1/3 next to the drawing B) I mean-well, that's what, okay, so this is what they said and their explanation was like you said, there are three things and one is shaded, so one out of three. And a lot of kids said that. So… that's not what you seem to be saying. Or do some people agree with that? Maybe some people here agree with that. Because that is what most people were saying. Does anyone agree-think they're right?

Despite the teacher's continuous efforts to persuade—but not to convince—and to find someone who agrees with the incorrect answer of 1/3, the students persistently keep their original answer. The EML 2009 cohort reaches a tacit collective agreement that 1/3 is not a correct answer for the second part of the brown rectangle problem, but the teacher further challenges the students to justify the reason why it is not called as 1/3.

Teri, who is another targeted student to explain in a public space, gets a floor to explain. Teri explains "Because they aren't equal" to the class. Even though Teri does not elaborate well, she provides the key idea that the teacher strives for and aims to elicit for developing a complete mathematical explanation for the brown rectangle problem. To specify the meaning of equal, the teacher probes further. The teacher invites Teri to come to the board to show parts that are not equal, but Teri declines the invitation to the board. After the teacher's several attempts, Teri makes a reference to a rectangle and two squares.

Teacher:	So what are you looking at up here that you think it isn't equal when you look at this diagram?
Teri:	The… rectangle.
Teacher:	Okay, so here? (pointing to the right side of the big rectangle) What is about this one?
Teri:	Because… that is a rectangle and on other side is, two squares.

With the teacher's further prompts, Teri elaborates the meaning of "not equal" but attributes "not equal" to different shapes—"a rectangle" and "two squares." After revoicing Teri's explanation, the teacher further challenges that squares are also rectangles—a special kind of rectangle—and asks for other reasons besides different shapes. As Marcellus expresses that he is lost, the teacher revoices Teri's explanation and further asks ideas about "not equal."

Teacher:	So Teri said these are not equal. And she pointed out this is a rectangle and these look like they're rectangles that we can call squares. But, something else also I think beside what Teri's already brought up. Something else makes them not equal too. What else do you see? What about Mannis?
Mannis:	They're not equal because all the third box-all the-they all need to be equal. They just all need to be equal.
Teacher:	Okay, so all-what do you mean by equal?
Mannis:	All the parts. All the-all the squares.
Teacher:	What about-what about them has to be equal?
Mannis:	They need to be equal- all the parts of the fraction.
Teacher:	Okay. So does somebody have a word for what Mannis and Teri are talking about? They have to be equal and in what way? So, people are saying that this isn't equal to these two. What's not equal about them?

Despite the teacher's continuous effort to elicit a word to describe "not equal" beyond different shapes, Mannis reiterates that all of the parts should be equal but does not elaborate the language what the teacher strives for. The teacher gives a turn to Marlais.

Marlais:	That the two-there's a rectangle and two squares. And they're-and the rectangle is not equal to two squares.
Teacher:	In what way, isn't it equal? What's not equal about it?
Marlais:	Isn't same size. Or shape.
Teacher:	Not the same size. It's actually, we could call that kind of size, area. This has a much bigger area than those two. Can you see that?
Marlais:	Yep.
Teacher:	Can everyone see that this has a bigger area?
Students:	Yes.
Teacher:	We want them all to have the same area.

After the teacher's continuous efforts, Marlais elaborates that the parts are not equal sizes. Using this term, the teacher introduces the term of "equal area." The teacher goes back to the first part of the problem to check whether all of the boxes have the same area and elaborates the tacitly agreed-on incomplete explanation for the first problem as "we can call this one one-third because there are three equal areas, and we have one of them out of the three." Pointing out that the second problem does not have three equal areas, the teacher asks what Tiara and Jacqueline did for the second problem and why it is important to draw a line.

Teacher: So, we can call this one one-third because there are three equal areas, and we have one of them out of the three. Here (pointing to the second problem), we don't have three equal areas. So what was it that-that Tiara and Jacqueline did over here? Why is it important what they did when they drew the line? Can someone explain that? Why was that important? Teri? Tonya?

Tonya: So they can all have equal parts.

Teacher: Say it a little louder.

Tonya: So they can all have equal parts.

Teacher: Good. That was loud enough. So, by drawing the line, they made the shapes so they could see all four parts that all have equal area. So, is that what you were doing, girls? Jessican and Tiara? Okay.

Tonya elaborates that drawing a lines makes equal parts for the second problem. She produced very interesting explanation in her notebook at the beginning of the lesson, but provides an accurate explanation at this point.

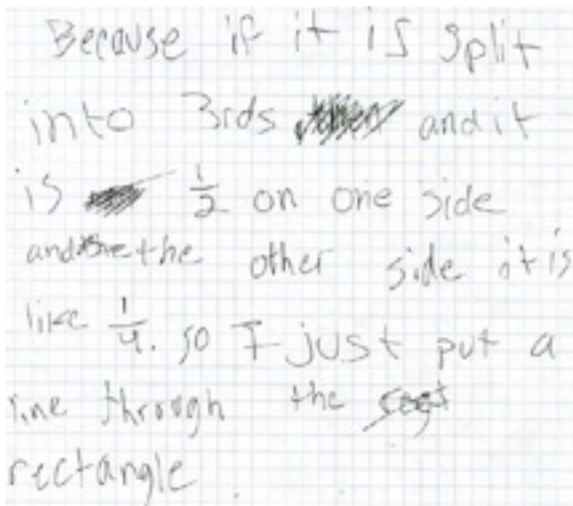


Figure 4.12. Tonya's written explanation for the second part of the brown rectangle problem

After eight-minute of whole-group discussion about the second problem, the teacher posts an additional problem, not an equally partitioned one, for students to work with a partner. A few minutes later, the teacher corrects her writing on the board, in which she wrote 1/3 next to the second problem, and asks for another fraction which is not shaded, and writes an idea of "equal area" on the board which is collectively developed during a whole-group discussion.

Summary

The EML 2009 students propose three answers (1/3, 2/3, and 2/6) for the first part of the brown rectangle problem and discuss two proposals (1/4 and 1/3) for the second part of the brown rectangle problem (the correct answer of 1/4 is proposed by the student and the incorrect answer of 1/3 is introduced by the teacher). In explaining the brown rectangle problem on Day 4, the EML 2009 students initially miss the key idea of "equal" for naming a fraction; use the pre-defined mathematical term without specifying the underlying concept behind the term (e.g., denominator, numerator); use the geometric-feature-free terms (e.g., "three boxes," "three," "three spaces" rather than "three squares"); use the inaccurate language (e.g., "the third box") which its intended meaning is different from the accepted mathematical definition; have lack of specificity beyond counting; and describe an action taken (e.g., "put a line") rather than the reasoning, but do not have much difficulties with explaining in a public space.

For the first part of the brown rectangle problem, Aiyana proposes the correct answer (1/3) but provides the incomplete explanation (i.e., missing "equal"). The teacher's seeking for different answers makes the students to improvise two other proposals, 2/3 by Sandra and 2/6 by Callie. For the second part of the brown rectangle problem, Jacqueline and Tiara explain the correct answer (1/4) but provide the incomplete explanation (i.e., missing "equal"). As no other comments and proposals are made, the teacher introduces the incorrect answer of 1/3 to students. After eliciting an explanation why someone might see 1/3 from Malik, the teacher looks for someone who agrees with 1/3 but all of the students have the robust understanding about the answer of 1/4 for the second part of the brown rectangle problem. As a reason for not calling it as 1/3, the key idea of "equal" is brought up by Teri and then further elaborated by Mannis and Marlais. Unlike the previous years, most of the EML 2009 students come up with the correct answer, have a strong conviction about the correct answer of 1/4 for the second problem, and build a robust understanding about the key idea of naming a fraction—making equal parts.

To support students' development of mathematical explanation for the brown rectangle problem, the teacher distributes an equal opportunity to explain rather than heavily controlling the sequence of proposals; seeks for different answers; makes an extensive use of the initial explanation rather than eliciting various versions of explanation and then just leaving them unexamined; request for repeating and revoicing the initial explanation; invites the students to the board so that they supplement their verbal explanation with pictorial representation; uses a partner to have an initial agreement on the answer and to practice an explanation with each other in a private space; produces a written explanation in advance; and increases the level of mathematical supports over time (no substantive mathematical supports, including clarifying what the rectangle refers to and remediating errors during the set-up stage and the individual work).

4.5. The Case of EML 2010

Preview

In the EML 2010, the brown rectangle problem is introduced on the second session of Day 2. For the first problem (naming a fraction for the equally partitioned rectangle), only one answer (1/3) is proposed by Macaulay. After providing an explanation on the board, Macaulay elicits comments from the audience. Jaclyn, who produced the same correct answer of 1/3 in her notebook, challenges Macaulay by asking what if there is no line in the drawing. After hearing both Macaulay's thinking (1/2) and Jaclyn's thinking (a quarter of a half) for Jaclyn's comment, the teacher checks for agreement with Macaulay's answer for the first problem and makes a transition to the second problem.

For the second problem (naming a fraction for the unequally partitioned rectangle), the first proposal (1/4) is made by Shar. In a similar way, the teacher asks Shar to elicit comments from the audience. Coretta, who produced the same correct answer of 1/4 in her notebook, challenges Shar by asking what if there is another line in the drawing. The teacher makes sure that all of the students could hear Coretta's comment and then asks the students about the difference between Shar's drawing and Coretta's drawing. Ahmed notices the different shapes, Karl notices the different number of rectangles, and Dahlia brings up the idea of "equal parts." The teacher then gives a turn back to Shar to respond to Coretta's comment. Shar explains that Coretta's drawing is still 1/4 without an additional line in it. The teacher then creates a private space for students to write down whether they agree or disagree with Shar's response to Coretta's drawing.

The teacher gives turn to students, moving back and forth between agreement and disagreement. In this process, Macaulay shows his disagreement with Shar's answer for the original second problem and proposes his answer of 1/3 for the original second problem. With Macaulay's disagreement with the original second problem, the teacher puts Coretta's idea to the side and puts Macaulay's proposal on the table. The teacher first gives a turn to Shar to explain her answer (1/4) and then give a turn to Macaulay to explain his answer (1/3). The teacher gives turns to other students, including Amani, Dahlia, Michael, Jaclyn, Eric, and Javonte, to develop an explanation why it is 1/4 for the original second problem and then returns to Macaulay to explain.

After reaching an agreement about the answer for the original second problem, the teacher goes back to Jaclyn's idea. Karina, Hala, and Terrence develop the explanation that it is not 1/2 because it does not have equal parts in it. Because of the time constraint, the teacher wraps up the discussion for the original first problem, the original second problem, and Jaclyn's idea, but leaves Coretta's idea for further discussion at a later point of the EML program.

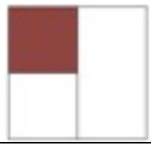
In the first session of Day 6, the teacher reviews the working definition of fraction that they have developed for the last few days and then revisits Coretta's idea. The teacher opens a whole-group discussion by checking how many students think that Coretta's drawing is 1/4 and how many students think that Coretta's drawing is not 1/4. Showing a disagreement with 1/4, Ahmed adds two additional horizontal lines to make equal parts and claims that the answer is 2/8. The teacher checks whether the students agree or disagree with Ahmed's idea. Showing a disagreement with Ahmed, Coretta suggests that there are two answers, either 1/4 or 2/8. The teacher invites the students to make comment on Coretta's explanation and Shelly comments that if adding another line, there are three answers. The extensive detailed analysis of 35-minute of instructional interactions in Day 2 and of 18-minute of instructional interactions in Day 6 for teaching the brown rectangle problem in the EML 2010 is provided below.

Extensive Detailed Analysis

After a brief introduction about the work that the students will do with fractions during the EML program, the teacher introduces the brown rectangle problem on the second session of Day 2 in the EML 2010. The teacher post two problems (the equally partitioned rectangle and the unequally partitioned rectangle) together, but each diagram has its own, but the same, written problem statement. Pointing out that there are two diagrams but the problem statements are the same for both diagrams, the teacher asks a volunteer to read aloud the problem. After Ella reads aloud the problem, the teacher asks the students to write down a fraction and to explain the reason for the answer in their notebooks, but does not clarify what "the rectangle" refers to at this point.

What fraction of the rectangle is shaded brown?

What fraction of the rectangle below is shaded brown?



While the students are working on the problem individually, the teacher circulates the classroom mainly to check whether the students write down an explanation about the answer they choose, but does not provide substantive mathematical support unless the students request for clarification. In her third circulation of the classroom, the teacher is engaged in mathematical conversations with two students, Coretta and Qayshawn. The first mathematical interaction begins with Coretta's request whether she can cut the right side of the big rectangle to make four pieces.

Coretta: Ms. Ball? Can I cut the half to make it to four pieces?
Teacher: Okay, so then explain what you did. You can do that if you explain what you did.

Coretta names the right side of the whole rectangle as "the half" instead of "the rectangle" or "the big rectangle" but does not clearly express what "it" refers to. The teacher neither extends a conversation with Coretta further (e.g., what "it" refers to, why she wants to cut the half, and what kind of pieces are produced after cutting) nor gives her a compliment about coming up with the idea that the teacher aims to elicit during the lesson. In responding to Coretta with a neutral tone, instead, the teacher encourages Coretta to elaborate on her reason for cutting it into four pieces.

The teacher continues to check whether other students write an explanation in their notebooks and then stops her walk in front of Qayshawn. The teacher encouraged Qayshawn to write an explanation during her second circulation of the classroom, but she notices that Qayshawn produced an explanation for the first problem but not for the second problem during her third circulation of the classroom. Similar to the comments that the teacher gave to other students, she begins a conversation with Qayshawn by encouraging him to write an explanation for the second problem. At the moment that the teacher stops her walk and stands in front of Qayshawn, he wrote 1/3 with an explanation "I onely [sic] one shaded in" for the first problem, but only wrote 1/4 without an explanation for the second problem and did not draws a line to make equal partitioning either.

Teacher:	Qayshawn, do you know why did you say that? How did you know that?
Qayshawn:	I counted.
Teacher:	What does that number say?
Qayshawn:	Huh?
Teacher:	What is that? What's this number?
Qayshawn:	One-fourth.
Teacher:	Okay, so how did you know that that's one-fourth? Can you write down how you know?
Qayshawn:	(silence)
Teacher:	How did you get that answer?
Qayshawn:	(silence)
Teacher:	Can you explain it to me? What did you-how did you figure out?
Qayshawn:	(silence)
Teacher:	How did you figure this one (pointing to the first problem) out?
Qayshawn:	I only see one shaded.
Teacher:	Okay, out of what?
Qayshawn:	All of them.
Teacher:	Okay. And how about this? (pointing to the second problem) There's one shaded.
Qayshawn:	Out of all of them.
Teacher:	So how do you know that that was one-fourth and the other one was one-third?
Qayshawn:	Because this one's shaded in out of all three of them. Out of all of them.
Teacher:	Right. But here (pointing to the first problem) you have one, two, three, and you said one-third. But here (pointing to the second problem) you-
Qayshawn:	It was one shaded in, but it's three boxes
Teacher:	And this one, how do you know this one is one-fourth?
Qayshawn:	(trying to change his answer) One three.
Teacher:	I didn't say that it was wrong. I'm just asking how do you know? It's actually right, but how did you know that?
Qayshawn:	It's only one shaded in…
Teacher:	Do you know how you came up with that?
Qayshawn:	It was only one shaded in.
Teacher:	What?

Qayshawn:	It was only one shaded in and one across right there would be four.
Teacher:	Okay, well do that. Show what you were thinking.
Qayshawn:	(starting to draw a line in his notebook)
Teacher:	Is that what you were thinking about? So put it in.
Qayshawn:	(drawing a line in his notebook)
Teacher:	Put that line in. Is that what helped you to decide? Okay, so say that, "because I put the line in."

The teacher initiates the conversation with Qayshawn to encourage him to write an explanation for the second problem. In responding to the teacher's initial request for explanation, Qayshawn says "I counted." His initial explanation neither portrays the idea about fraction nor corresponds with the answer he produces. The teacher continuously attempts to support him to build a correspondence between his explanation and the number he choose, but Qayshawn keeps his silence. Observing that Qayshawn has struggled with producing an explanation for the second problem, the teacher makes a transition to the first problem and asks how he figures out that the answer is 1/3 for the first problem. Breaking his silence, Qayshawn adds a little bit of details ("one shaded" and "all of them") but does not further specify the total number of parts in a whole for the first problem. After eliciting his explanation for the first problem, the teacher initiates an explanation for the second problem by saying "one shaded" and Qayshawn takes over the turn by adding "all of them." Qayshawn writes 1/3 for the first problem and 1/4 for the second problem, but produces the same explanations for both problems by saying "one shaded out of all of them."

Not being satisfied with the general explanation that Qayshawn produces, the teacher further challenges Qayshawn to elaborate his explanation. Qayshawn elaborates his initial explanation for the first problem by adding the information about the total number of parts in a whole ("three boxes"), but he still could not provide an explanation for the second problem. Rather, with the teacher's continuous challenges, Qayshawn attempts to change his correct answer of 1/4 to incorrect answer of 1/3. The teacher confirms that his initial answer is right and helps him explain how he comes up with 1/4 by drawing a line on the right side of the rectangle.

This short two-minute exchange between the teacher and Qayshawn in a private space illustrates how discourse resources and collective resources could be used to support students' development of mathematical explanation in a whole-group setting. I will elaborate on the details of these features in a later section, but briefly describe some here. The teacher begins the conversation with Qayshawn to encourage him to write an explanation in his notebook and to support the development of his partial (i.e., only provided information about the numerator, "one shaded") and general explanation (i.e., repeated the information about "all of them"). Despite the teacher's sustained supports, Qayshawn seems to get an impression that he might be wrong because of the continuous challenges by the teacher in a one-on-one context. Under the absence of collective resources that Qayshawn uses by hearing other students' repeating his explanation or receiving questions from his peers, it might be difficult for him to elaborate his partial and general explanation in a one-on-one interaction with the teacher. In addition, the often-used discourse moves to support the development of explanation such as "Could you repeat?" is not used in a one-on-one interaction between the teacher and Qayshawn. In a whole-group setting, repeating the initial explanation is not requested by the teacher's own needs to hear the explanation better, but functions to support the audience to understand the proposed explanation better.

After five-minutes of individual work, the teacher convenes the class for a whole-group discussion. As the teacher invites a student to volunteer to come to the board to explain, seven students (Ella, Shelly, Jaclyn, Shar, Macaulay, Dahlia, and Thailee) raise their hands to volunteer to explain. The selection of an initial explainer is made through an effort to distribute an equal opportunity to come up to the board to explain, rather than based on the teacher's observation about the level of completeness, elaboration, and sophistication of mathematical explanation that the students produced in their notebooks or with the teacher's intention of sequencing solutions in a particular order. The teacher gives a turn to Macaulay. While Macaulay makes his way to the board, the teacher sets the expectation both for the explainer (speaking nice and loud) and for the audience (paying a close attention to the explainer).

Macaulay:	I-I picked-and it says, what fraction of the rectangles is shaded brown? And I-and I- my answer was one third of the-[of the-]
Teacher:	[You can use a marker] and write that number.
Macaulay:	(writing 1/3 on the board) I put one-third of the rectangle that's shaded brown.
Teacher:	Can you explain why you came up with that?

Macaulay:	Well because, there is three squares and one of them is colored in. So you pick the one-you write the number that is like the… amount of that is shaded in, then you write the whole entire numbers without all the squares in it.
Teacher:	So now you can say that "does anybody have comments?" You can say that to the class.
Macaulay:	Anybody get comments?

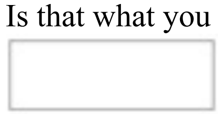
After reading the problem statement, Macaulay explains his reasoning about why he gets 1/3 for the first problem. Macaulay speaks loud enough for other student to hear his explanation, provides an explanation facing toward the audience, and describes both the part (one of them is colored in) and the total number of parts in a whole (three squares), but misses the key idea for naming a fraction ("equal"). Adding more details to his explanation, he makes it unclear what "the whole entire numbers without all the squares in it" means. Unlike the previous years when the teacher requested students to repeat the explanation or made her own attempt to elicit comments from students, the teacher empowers Macaulay to elicit comments from the audience. Jaclyn raises her hand to give a comment to Macaulay.

Jaclyn:	Another way you can find out is… by... instead of just saying there is three, you can count them, and you can count just one 'cause one is just shaded in.
Macaulay:	You can do that too.
Jaclyn:	Like… for example, what if there was no picture?
Teacher:	If there was no picture?
Jaclyn:	Then how would you know if there was three squares?
Teacher:	So what-how then would you decide?
Jaclyn:	By then like counting… Umm…
Macaulay:	Because if it has no lines through it, it's a one whole.

It is not easy to figure out what Jaclyn offers as a comment at first, but seems that Jaclyn addresses the need of "counting" more explicitly. At this point, reviewing Macaulay's written explanation and Jaclyn's written explanation in their notebooks during individual work might be helpful to understand this conversation. A minor difference is observed between Macaulay's written explanation and Jaclyn's written explanation: Macaulay wrote "I know because 1 is shaded and there are 3 square" whereas Jaclyn wrote "I know by counting the squares." Both of them produce the correct answer (1/3), but neither of them provides the complete explanation for the first problem in their notebooks. It is interesting to observe that both pay attention to the shape (squares), but Macaulay uses "stative verb" (i.e., "are") and Jaclyn uses "action verb" (i.e., "counting") to describe squares. It is unknown how Jaclyn perceives the difference between stative verb ("are three squares") and action verb ("counting the squares") mathematically, but the minor difference of language choice seems to serve as an impetus for Jaclyn to give a comment to Macaulay.

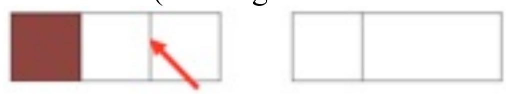
Not being perplexed by Jaclyn's challenge, Macaulay provides an immediate response to Jaclyn's comment. As Macaulay simply acknowledges Jaclyn's comment but does not consider it seriously, Jaclyn further challenges Macaulay with another example to convince him of her idea. It is unclear what Jaclyn refers as "no picture" but Macaulay interprets it as "no lines" without having any difficulties. Again, it is interesting to observe that Jaclyn uses the stative verb "was" in challenging Macaulay but uses the action verb "counting" in describing her idea. Even though the teacher empowers Macaulay to elicit comments at the beginning, she does not just watch the conversation unfolded but intervenes to unpack Jaclyn' mathematical idea in a public space. Macaulay provides his answer ("one whole") to Jaclyn's comment, but the teacher hears the mathematical need brought up by Jaclyn and attempts to articulate Jaclyn's idea first.

Teacher: Do you wanna come up to the board and draw what you're talking about, Jaclyn and show us?
Jaclyn: (Shaking her head) No.
Teacher: No? So are you saying if there's no lines in the drawing?
Jaclyna: Yes.
Teacher: So are you talking like-like what Ja-what Macaulay just said? Like a rectangle with no lines in it?
Jaclyn: Yes.
Teacher: Can I draw and see this is what you mean? May I draw something? So if we had one, I'll try to make it similar to what we have there. Is that what you mean?



Jaclyn: No.
Teacher: Come on up.
Jaclyn: (coming to the board)
Taecher: Is everyone trying to figure out what Jaclyn's talking about? [Okay, Okay.]

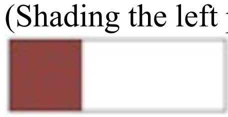
Jaclyn: [It's sort of like this, like-] there was no lines through here (pointing to the line in the first problem), but it's like one line where it is. (drawing a line in her drawing) Like that.



The teacher invites Jaclyn to the board to express her idea, but Jaclyn declines the invitation at first. The teacher then serves as a delegate to fully address Jaclyn's idea in a public space. The teacher translates Jaclyn's initial comment "if there was no picture in the drawing?" to "if there's no lines in the drawing?" based on Macaulay's restatement and draws a diagram based on what Macaulay interprets. As the diagram that the teacher drew does not portray Jaclyn's intention, because of the lack of specificity in Jaclyn's comment, Jaclyn, who initially declines the invitation to the board, naturally comes up to the board to explain her idea.

Teacher: So then, what would you say if it was like that?
Jaclyn: I was asking him. I don't know.
Macaulay: Oh. And that part (pointing to the left side of the rectangle) was shaded in?

Jaclyn: Yes.
Macaulay: So, and that would be one, one, one… half. Like one is shaded in and the other half is not.
Jaclyn: So you mean like one-third of the whole?
Macaulay: Uh, huh.
Jaclyn: Okay. (putting down the marker and coming back to her seat)
Teacher: So, what did you say, Macaulay?
Macaulay: So, if this is shaded in, one of them is still shaded in, it's still two umm, two things, one two.
Teacher: (Shading the left piece) Okay.



Macaulay: So, you have to draw one and then a two.
Teacher: So, you're saying then it would be one-half?
Macaulay: Yup.
Teacher: Is that what you think, Jaclyn? Then it's one-half?
Jaclyn: No.
Teacher: You don't think it's one-half? What do you think?
Jaclyn: It's like a quarter of a half.

Teacher:	It's a quarter of a half?
Jaclyn:	Yes.
Macaulay:	[…] oops.
Teacher:	[So…]
Jaclyn:	[Like], on that picture (pointing to the board from her seat), there are three squares…
Teacher:	So, let's-let's talk about this one after we do this one (pointing to the second problem). And then we can go back to your example. Is that okay? Do people agree with Macaulay that this one is one-third the way this one is drawn? Did anyone have something different? Okay. Thank you Macaulay, for coming up. Who would like to explain the second one?

After Jaclyn draws her idea on the board, the teacher asks Jaclyn what she thinks about, but Jaclyn tosses the turn to Macaulay. Macaulay explains that it is 1/2 because one is shaded and the other half is not shaded, but Jaclyn mishears Macaulay's explanation as 1/3 and Macaulay confirms what Jaclyn misheard. Not overlooking the moment of mishearing between Macaulay and Jaclyn, the teacher asks Macaulay about his thinking first and then checks with Jaclyn about her thinking.

Even though Macaulay provides an incorrect answer to Jaclyn's drawing, he makes a reasonable mistake which is consistent with his reasoning for naming a fraction (he also writes 1/3 for the second problem). It is interesting to observe that Macaulay chooses the language of geometric names "three squares" to describe the equally partitioned rectangle for the original first problem but chooses the language of geometric-feature-free names (e.g., "one shaded in and the other half is not"; "two things") to describe the unequally partitioned rectangle problem for Jaclyn's example with lots of stuttering.

On the other hand, it is difficult to figure out the logic underlying Jaclyn's answer for "a quarter of a half." The teacher does not further probe where "a quarter of a half" comes from. One reason might be that unpacking "a quarter of a half" does not contribute to develop the key idea of "equal" for naming a fraction and the students do not establish the knowledge of figuring out "a fraction of a fraction" to resolve the issue that Jaclyn faces. When Jaclyn tries to explain her answer further by pointing out "three squares," the teacher interrupts Jaclyn's further explanation, confirms the agreement about the answer (1/3) for the first problem, and then delays the discussion about Jaclyn's idea until they discuss the second problem.

The example that Jaclyn proposed has the same mathematical feature as the second problem—unequally partitioned rectangle—but comparing the original first problem (equally partitioned rectangle into three equal pieces) and Jaclyn's example (unequally partitioned rectangle into two different size of pieces) requires different mathematical work for the teacher to support the development of mathematical explanation than comparing the original first problem (equally partitioned rectangle into three equal pieces) and the original second problem (unequally partitioned rectangle into three different size of pieces).

In this short exchange between Macaulay and Jaclyn, with the intervention by the teacher, several noticeable mathematical works are observed. First, the role of proposer, explainer, audience, and commentator[18] is not strictly determined. At the beginning, Macaulay proposes his solution of 1/3 and explains his answer. Jaclyn plays a role of commentator about missing the word "counting," but manipulates the key variable of the brown rectangle problem to back up her idea. Later, Macaulay responds to Jaclyn's proposal. As they exchange mathematical ideas, they have switched their roles over time. Second, Jaclyn is unwilling to come to the board to draw her idea on the board at the beginning, but naturally comes up to the board after the teacher serves as a delegate to make Jaclyn's idea in a public space. Third, it seems that Macaulay and Jaclyn exchange conversation smoothly, but they actually mishear each other. Not overlooking this moment of mishearing, the teacher checks back with Macaulay and Jaclyn in the ongoing conversation.

After checking for agreement on the answer of 1/3 for the first problem, the teacher calls for other proposals for the first problem. All but two students, Mustafa and Terrence, write 1/3 for the first problem in their notebooks. Mustafa wrote 1/2 and Terrence wrote 3/1. It is not clear how Mustafa initially gets 1/2 based on his explanation ("if you count the boxes and one is shaded in"), but one possible reason might be that he

[18] I purposely use the word "commentator" instead of "commenter." Considering that the difference between "comment" (an isolated remark) and "commentary" (a series of remarks, explanations, and interpretations), it is more appropriate to say that Jaclyn plays a role of "commentator" instead of "commenter."

counts the number of shaded box over the number of unshaded boxes rather than counting the number of shaded box over the total number of boxes in a whole. Terrence just switches the number that should go into the numerator with the number that should go into the denominator, and get 3/1 instead of 1/3. He wrote “I know because there are 3 boxes so the 3 should be on top and the part that is shaded with is 1 one is shaded so the 1 should be on the bottom” in his notebook. In describing the fractional parts, he uses the terms such as “top” and “bottom” instead of “numerator” and “denominator.” Despite the fact that most of students get the correct answer (1/3) for the first problem, nobody indicates “equal” for explaining the answer in their notebooks.

As no one makes further proposals for the first problem, the teacher moves on to the second problem. The teacher makes an effort to distribute turns equally to students who have not had a chance to come up to the board rather than heavily controlling the sequence of the answers. For the second problem, 17 students (Anthony, Bernard, Chanika, Coretta, Dahlia, Eric, Ella, Jaclyn, Karina, Kassandra, Karl, Michael, Qayshawn, Shar, Shelly, Thailee, and Zahara) wrote 1/4, nine students (Elias, Hala, Javonte, Jason, Macaulay, Madeline, Mustafa, Ahmed, and Samara) wrote 1/3, and one student (Terrence) wrote 4/1 in their notebooks[19]. Among them, four students (Anthony, Dahlia, Michael, and Zahara) mention “equal” in their written explanations.

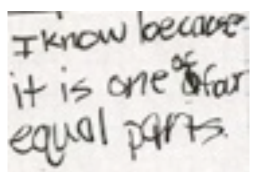


Figure 4.13. Anthony’s written explanation for the second problem of the brown rectangle problem

[19] Devante was absent for the class in Day 2.

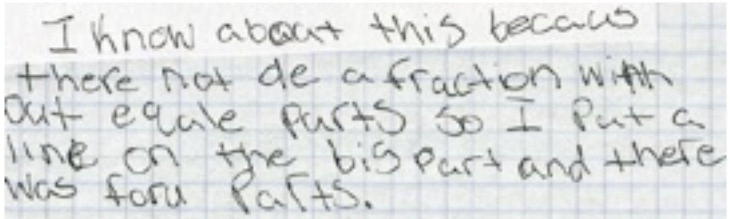


Figure 4.14. Dahlia’s written explanation for the second problem of the brown rectangle problem

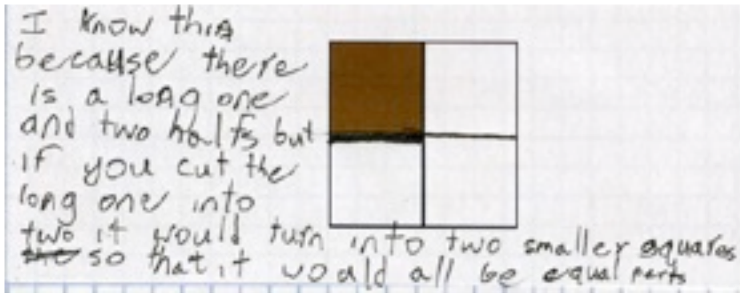


Figure 4.15. Michael’s written explanation for the second problem of the brown rectangle problem

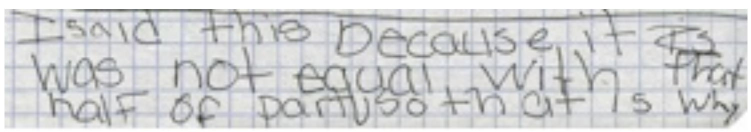


Figure 4.16. Zahara’s written explanation for the second problem of the brown rectangle problem

Several students, including Karina, Shar, Bernard, and Thailee[20], raise their hands to propose an answer for the second problem. As shown in Figures 4.17 through 4.20, they have the same answer (1/4) and produce mathematically similar level of written explanations in their notebooks.

[20] Because of the camera angel, I am able to identify only four students’ names.

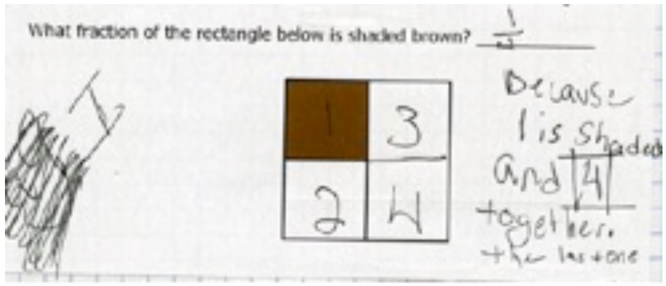


Figure 4.17. Karina's written explanation for the second problem of the brown rectangle problem

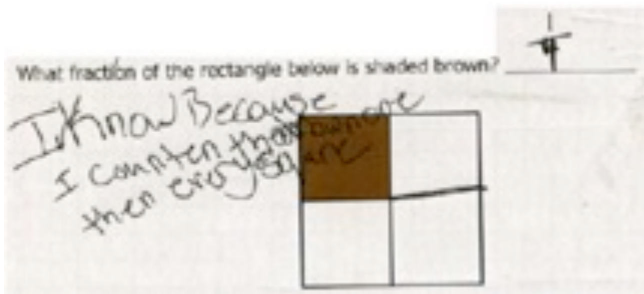


Figure 4.18. Shar's written explanation for the second problem of the brown rectangle problem

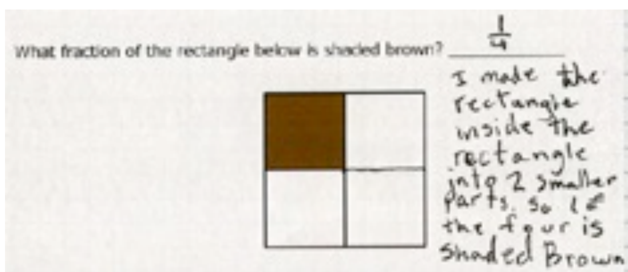


Figure 4.19. Bernard's written explanation for the second problem of the brown rectangle problem

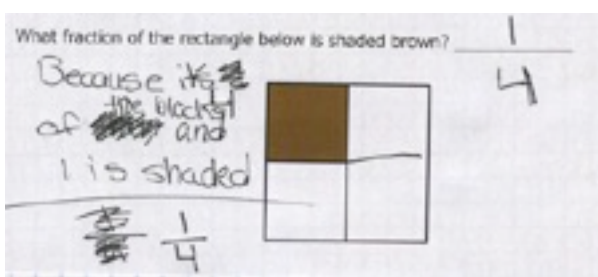


Figure 4.20. Thailee's written explanation for the second problem of the brown rectangle problem

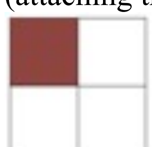
After checking with Shar whether she has gotten a chance to come to the board, the teacher gives a turn to Shar to explain her proposal for the second problem. While Shar makes her way to the board, the teacher praises the students' good watching and listening to the explainer and asks them to listen carefully to Shar's explanation because Shar will ask comments after her explanation. After Shar writes her answer of 1/4 on the board, the teacher asks Shar to explain her answer. Shar volunteers to explain on the board, but she shakes her head with the request to explain, keeps her silence, looks at the drawing on the board, and fiddles with the marker that she used to write the answer on the board. At the teacher's second attempt to elicit an explanation, Shar begins her explanation.

Teacher: What were you thinking about when you wrote that? How did you look at the picture and decide what you write? Can you tell us what you were thinking?

Shar: That this (pointing to the right side) can be two squares here.

Teacher: Here. Here's something you can use to show us. I think this is- maybe will help you. This is a line you can attach to the drawing. Why don't you put the line what you were thinking about it?

Shar: (attaching the line in the middle of right side area)



Teacher: Is it what you wanted?

Shar: Uh-huh.

Teacher: Could now-now can you explain to the class what you were thinking?

Shar: I was looking that this (pointing to the left side of the rectangle) one was shaded in and there was- this rectangle (pointing to the right side of the rectangle) was two squares. So, it was one-fourth.

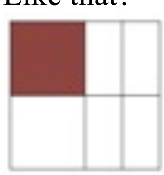
Teacher: And why did you get-where did you get the four?

Shar: I counted all these.

Teacher: Okay, now you can say 'comments.'

Shar: Any comments?

Shar volunteers to explain her answer on the board, but refuses to give an explanation after writing her answer on the board at the beginning. Faced with Shar's shyness, hesitance, and refusal to give an explanation, the teacher rephrases her question in a way that allows Shar to build a correspondence between the answer and the pictorial representation. After the teacher makes several attempts to elicit an explanation, Shar begins her explanation by pointing out the right side of the rectangle and explains what she gets (two squares) but skips several steps (adding a line makes a rectangle into two squares; adding a line makes equal parts; and there are four equal parts) in her explanation. Hearing Shar's mathematical needs, the teacher hands over the sticky line to Shar so that Shar clearly visualizes the line added. Shar adds more details in her verbal explanation than her written explanation, but does not make explicit that she adds a line verbally, why she adds a line, and where she gets four parts. Similar to what the teacher did with Macaulay, the teacher opens a whole-group discussion by empowering Shar to elicit comments from the audience. Coretta raises her hand to provide a comment.

Coretta:	Um, when you cut the square half, what if you would cut it in half again? How many would you have? Like if you could have split down in the middle?
Teacher:	Do you mean if we were to do…
Shar:	(drawing a vertical line with her finger on the right side of the big rectangle) Like that?
Teacher:	(drawing a vertical line with her finger on the right side of the big rectangle) Like this?
Coretta:	Yes.
Teacher:	Let's get another one up there and try what Coretta's saying (putting up another copy of the original drawing for the second problem). Let's see. I thought maybe we could use these to put it up. Do you think she-you know what she's asking about?
Shar:	(nodding her head) I think…
Teacher:	Can you draw what you think Coretta-can you draw-tell her what you want her to draw?
Coretta:	Um, not going this way, but going up and down.
Teacher:	Both? Both lines?
Coretta:	Yeah.
Teacher:	Okay. So the one you did and another one, I think. Ask her if that's what she means.
Shar:	Like that? 
Coretta:	Yeah. How is that-would it still be the same thing? Or would you have more?

Teacher: That is a really, really, really good question, Coretta. Why is that a good question what Coretta was asking? Can someone-before we try to answer it, what is that such a good question?

The exchange between Shar and Coretta for the second problem has very similar mathematical structure to the exchange between Macaulay and Jaclyn for the first problem. For the first problem, Macaulay presents his correct answer of 1/3 with his incomplete explanation. Jaclyn, who also wrote the same answer of 1/3 for the first problem in her notebook, shows her understanding of Macaulay's explanation ("instead of just saying that there is three") and then challenges his explanation with "what if" question instead of just showing her agreement with Macaulay's answer. Jaclyn did not have a clear answer for her question at first, as shown in her prediction that the answer might be a quarter of a half, but shares her mathematical observation.

For the second problem, Shar also presents her correct answer of 1/4 with her incomplete explanation. Coretta, who also wrote the same answer of 1/4 for the second problem in her notebook, shows her understanding of Shar's explanation ("when you cut the square half") and then challenges Shar's explanation with "what if" question, instead of just showing her agreement with Shar's answer. As shown in the later conversation, saying either 1/4 or 1/6, Coretta also does not have a clear answer for her comment at this point, but shares her mathematical observation.

Through the "what if" question, which is very important mathematical practice, Jaclyn raises the important mathematical issue of removing the existing line that results in unequal partitioning and Coretta raises the important mathematical issue of adding an additional line that results in unequal partitioning. In providing the comments, Jaclyn and Coretta are based on the same, but incorrect, mathematical reasoning that removing or adding a line changes the answer, which is mutually exclusive. Jaclyn wrote 1/3 for the first problem in her notebook, but predicts that the answer for her drawing would be a quarter of half. Mathematically, both 1/3 and 1/8 could not simultaneously be considered as an answer for naming the same fractional amount. Similarly, Coretta wrote 1/4 for the second problem in her notebook, but predicts that the answer for her drawing would be either 1/4 or 1/6, which are mutually exclusive. Mathematically, both 1/4 and 1/6 could not simultaneously be considered as an answer for naming the same fractional amount.

In addition, for both cases, Macaulay and Shar are not perplexed with that their correct answers are challenged, but make efforts to hear, understand, and respond to the comment.

Instead of just watching how the conversation unfolds, the teacher intervenes to clarify whether Coretta adds two lines (both the horizontal line and the vertical line) from the original drawing and pauses to make sure that everyone fully understands Coretta's comment. It is not a private conversation between Shar and Coretta, so the teacher tries to make sure that all of the students understand the mathematical issue raised before Shar provides an answer to Coretta's comment. Putting Coretta's idea on the table, the teacher asks whether there are any differences between Shar's drawing and Coretta's drawing. Ahmed gets a turn and explains.

Ahmed:	Because the one Shar made, like it's a whole square, but these ones are like, they are kinds of like mini rectangles.
Teacher:	They are like mini rectangles? Are these the same whole? Is this (Shar's drawing) the same whole? This is the whole rectangle. Do both of these have (pointing to Coretta's drawing) the same whole?
Students:	Yes.
Teacher:	They have the same whole. The rectangles is the same whole.
Shar:	It's the same thing as that one (pointing to 1/4).
Teacher:	But, what's.. okay, wait before we get to that. You are already thinking about her question. What's the difference between the first drawing and the second drawing? Can you see, Karina and Jason? I think-Shar, why don't you stand where I'm standing? Cause you can still see a little bit and everyone could see. What's the difference between the first drawing over there and the second drawing that Coretta suggested? They both had same whole rectangle, but what's the difference between them? Karl?
Karl:	There's more um… more… more rectangles in one of them and [there's only two-].
Teacher:	[Okay there are more] rectangles divided up in this on. What else is different? What else is different between the two? Something else important.

Ahmed initially wrote 1/3 for the second problem in his notebook. He does not fully articulate his explanation and uses the language of "a whole" obscurely, but is now able to see different shapes beyond just counting the number of pieces. Instead of further probing Ahmed's explanation, the teacher clarifies whether Shar's drawing and Coretta's drawing have the same whole, which is another important idea for naming a fraction.

One reason might be that Ahmed uses the language “a whole” differently than the intended and accepted meaning for naming a fraction, thus clarifying “what the whole is” would contribute to the development of mathematical explanation. Another reason might be that the same whole needs to be premised to compare fractions.

Even before the teacher finishes her talk, Shar chimes in and then attempts to provide her answer for Coretta’s drawing. Not letting Shar continue to blurt out the answer for Coretta’s question, the teacher returns to her original question of figuring out the difference between Shar’s drawing and Coretta’s drawing again. Next, Karl gets a turn to explain the difference, but he explains the difference in the number of rectangles. The teacher has a quick repeat of Karl’s idea that the number of rectangles is different between Shar’s drawing and Coretta’s drawing, but she stays focused on figuring out another difference between Shar’s drawing and Coretta’s drawing. Dahlia gets a turn to explain.

Dahlia:	They’re not equal parts.
Teacher:	They are not equal parts in the second one. How do you know?
Dahlia:	Because I learned from my fourth grade class.
Teacher:	So can you see these aren’t all equal split in this one (pointing to Coretta’s drawing)? What about the one that Shar was drawing? Are the parts equal in that one?
Dahlia:	Yeah.
Teacher:	Do you agree with that-
Dahlia:	Yeah, just-
Teacher:	They’re the same as the brown-the brown one. Are all the same? And what-and this one (pointing to Coretta’s drawing) are all the same in size as the brown one?
Dahlia:	No.
Teacher:	Does everyone agree with that? That they’re not all the same size in the second one? Okay.

Dahlia, who explicitly laid out the idea of “equal” in her notebook, brings up the idea of “equal parts” to the whole-class. In responding to the teacher’s question about the basis of her reasoning, Dahlia grounds her response on non-mathematical reasoning—what her fourth grader teacher told her. The teacher does not take up Dahlia’s non-mathematical reasoning, but applies Dahlia’s idea of “equal” to Shar’s drawing and Coretta’s drawing. After checking with the agreement about Macaulay’s answer of 1/3 for the first problem, this is the next moment that the teacher explicitly asks for an agreement from students. The teacher first ask for agreement from Dahlia and then asks for agreement from the whole-class about the idea that Shar's drawing has equal parts but Coretta's drawing does not have equal parts. After clarifying the difference between Shar's drawing and Coretta's drawing, the teacher gives a turn back to Shar to explain her answer for Coretta's question.

Shar:	That one-fourth is the same about this one because all it is a split down there, but it's the same thing as that one, but it just has a split right there.
Teacher:	Okay, can people hear Shar?
Students:	No.
Teacher:	Could you try it one more time? 'Cause it's very interesting what you are saying, but not everybody can hear you. Talk really clearly.
Shar:	That it's the same thing as this one, because it just has a split in the middle and it's the same drawing as this one, but it just.. so it's the same answer like this one.
Teacher:	So I-can I try repeating it? So, in this one, you say it's one-fourth, because why? In the first one? Why is this one-fourth here?
Shar:	Because it's one colored in, and it's four out of these.
Teacher:	Four of those, though using Ja-Ja-Dahlia's idea, they are all what?
Shar:	They are all squares.
Teacher:	They are all squares and what else Jaclyn? Dahlia? In this drawing, they're all- what did you say?
Dahlia:	Squares.
Teacher:	And there're all what?
Dahlia:	They are all equal parts.
Teacher:	They are all equal parts. So, they're one-it's divided into four equal parts, and one of them is shaded. So, we call that one-fourth because there are- the whole is divided into four equal parts and one of them is shaded. This one is very surprising that Coretta suggested we draw, because it doesn't have all equal parts. But this, I think what Shar is saying. So can everybody listen carefully and you tell me if I'm saying it correctly. If you look at this, you can still see that it's divided into, let's make it really dark here. There's one division and there's the other one. So if you just look at those, you can see one, two, three, four. Without paying attention to that other line, you can still see four equal parts. One, two, three, four, and she's saying it's still one-fourth of the whole.

Shar was too shy to provide an explanation on the board at the beginning, but actively participates in hearing, interpreting, and responding to Coretta's comment. Without any hesitation at this time, Shar gives an explanation why she thinks that it is still 1/4 for Coretta's drawing. Shar speaks toward the board, hastily pointing to the pieces with a marker, so it is not easy for other students to track her explanation. The teacher checks whether other students understand Shar's explanation and then asks Shar to repeat her explanation. Even for her repeated explanation, Shar does not provide sufficient evidence of her claim beyond repeating the information that there is another line in Coretta's drawing. As Shar struggles with explaining why it is still 1/4 for Coretta's drawing despite her conviction, the teacher goes back to the original second problem to make clear why it is 1/4 and supports Shar to use the idea of "equal" that Dahlia brought up to justify the answer. Shar still stays with the shape, but the teacher makes a reference to Dahlia and makes it clear about equal parts. To make clear that Coretta's drawing has the same equal parts, the teacher asks students not to pay attention to the line, but to the four equal parts.

Teacher: Write in your notebook if you agree or disagree. "Is it still one-fourth in both of them?" So, what Dahlia is saying is it has to be in equal parts and Shar is saying you can still see the equal parts here. One, two, three, four. You don't have to pay attention to those. So see if you agree with that or don't agree with that. Can you see the four equal parts in the second one and then brown is still one fourth or not. Is that right? Is that what you are saying?

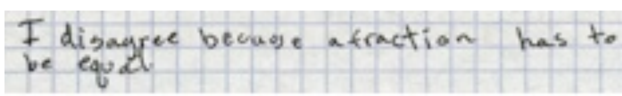
This is the third moment that the teacher asks students whether they agree or disagree with the idea. After the brief discussion about Coretta's drawing, the teacher creates a private space for the students to think whether they agree or disagree with Shar's proposal of 1/4 for Coretta's drawing. In the notebook, 16 students express their agreements with naming Coretta's drawing as 1/4, four students express their disagreements with calling Coretta's drawing as 1/4, and eight students do not have any records about their ideas.

Table 4.3. The records of agreement and disagreement with Shar's answer for Coretta's proposal

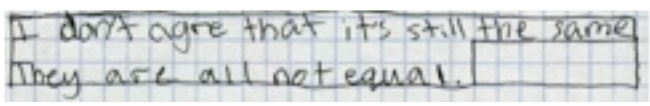
Agreement	Students' name
Agree	Anthony, Ahmed, Karina, Coretta, Ella, Hala, Jaclyn, Jason, Karina, Mohamed, Qayshawn, Samara, Shelly, Terrence, Zahara, Madeline*
Disagree	Bernard, Eric, Kassandra, Michael
No written records	Dahlia, Karl, Chanika, Elias, Javonte, Macaulay, Shar, Thailee

As a reason of disagreement, all of the four students raise an issue that parts need to be equal to name a fraction, even though they chose different language to describe it (see Figure 4.21). It is interesting to observe that all of the four students who disagree with naming Coretta's proposal as 1/4 produces the correct answer for the original second problem.

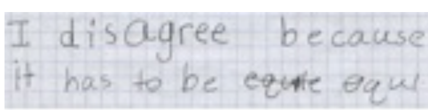
Bernard:


I disagree becuse afraction has to
be equal

Eric:


I don't agre that it's still the same
They are all not equal.

Kassandra:


I disagree because
it has to be ~~equte~~ equl

Michael:

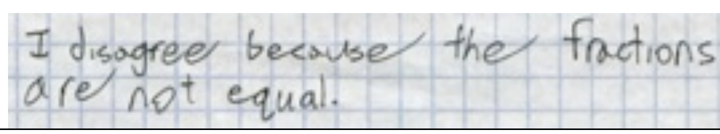

I disagree because the fractions
are not equal.

Figure 4.21. Reasons for disagreement with calling Coretta's drawing as 1/4 in the notebooks

Among 16 students who express their agreements about naming Coretta's drawing as 1/4, only four students wrote the detailed reasons for their agreements, but the reasons for agreements vary (see Figure 4.22) and most of students briefly indicate their agreements without providing reasons for the agreement.

Anthony:	I agree Jasmin The split dosen't matter the still 4 equal parts if you put them together
Hala:	I agree that it does not have to be ~~eqle~~ eqal.
Shelly:	I agree because $\frac{2}{8}$ are the same as $\frac{1}{4}$
Madeline:	I agree with that Because if there unecul or split it ~~[illegible]~~ wold not be Four

Figure 4.22. Reasons for agreement for calling Coretta's drawing as 1/4 in the notebooks

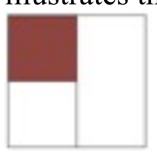
Anthony, Hala, Shelly, and Madeline provide reasons for their agreements in their notebooks, but the mathematical rationales are quite different. In fact, Hala's reason for agreement is mathematically incorrect because her claim contradicts with the definition of fraction. Anthony foregrounds the idea of equal parts, while considers the additional split line as background, which exactly matches with the teacher's last comment, "Without paying attention to that other line, you can still see four equal parts." Shelly imagines adding additional line in Coretta's drawing to make all equal parts and advances to the idea of equivalent fraction. Even though Madeline shows agreement, her explanation indicates that it is not four if it is unequally partitioned. After a few seconds later, the teacher asks students to read aloud what they wrote in the notebook. Thailee, Bernard, and Coretta explain whether they agree or disagree.

Thailee:	I agree. Because um, if they are unequal and they're split, then what I learned is that you could kind of like, make them one whole.
Bernard:	I disagree because a fraction always has to be equal. And what-what Shar is trying to say is that you can still see the fraction, I can see it, but it wouldn't be one-fourth because you have that line through the middle
Coretta:	If you have a line in the middle, you could say either it's… you could say it's either one-sixth cause now you have six pieces, but

still four. I also agree that if you have four pieces instead of six, so you could actually do it either way.

Three points of view are shared: (1) the answer is 1/4; (2) the answer is not 1/4; and (3) the answer is either 1/4 or not 1/4 (1/6). These comments are mathematically very rich. Coretta's idea, either 1/4 or not 1/4, is mutually exclusive one. Instead of delving into each comment, the teacher turns to another student, Macaulay, who wants to share his idea. Macaulay shows his disagreement, but it is for the answer of 1/4 for Shar's drawing, not for the answer of 1/4 for Coretta's drawing.

Macaulay: I disagree.
Teacher: With whom do you disagree?
Macaulay: I disagree with that.
Teacher: Okay.
Macaulay: Because it doesn't have a line through it, so that's just three.
Teacher: Like this (removing the sticky line that Shar attached which illustrates the second problem)?

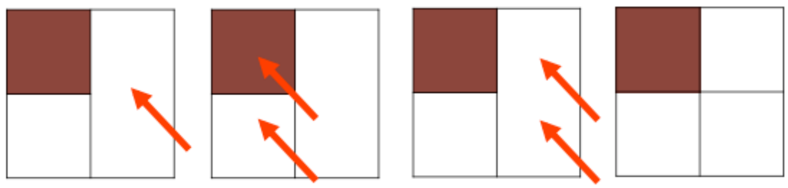


Macaulay: Yes.
Teacher: Back to the original one? Okay. So, let's go back to this for a minute. So, you're disagreeing with that. So, what are you thinking the answer is here?
Macaulay: I think that the answer is one third, the same as the top one.
Teacher: So, let's put this problem as- Coretta, I'm gonna put your problem aside for a moment.

Macaulay, who wrote 1/3 for the second problem in his notebook, shows his disagreement with Shar's answer of 1/4 for the second problem. Two mathematical features are observed. First, to validate his answer of 1/3 for the second problem, Macaulay makes a reference to the first problem because both diagrams have three parts and the actual size of the brown rectangle is the same. In the EML 2007 and the EML 2008, the teacher made a reference to the first problem when the incorrect answer was proposed or introduced. Either by the teacher or by the students, making a reference to the first problem while introducing the incorrect answer of 1/3 is used to validate the claim. Second, Macaulay uses the language "three squares" for explaining the first problem, but uses the language "three" for explaining the second problem.

The mathematical decision about the answer for Coretta's drawing could be built on the agreement about the answer for Shar's drawing, so the teacher puts aside Coretta's idea for a minute and then returns to the original second drawing. Before giving a turn to Macaulay, the teacher first asks Shar to explain again why the answer for the second question is 1/4.

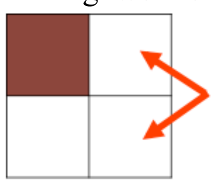
Shar: It's one-fourth because this is a rectangle (as shown in a) and these are all squares (as shown in b) and there's like these two rectangles and this-two squares (as shown in c) in this one, and I (attaching the line as shown in d)-and then…



(a) (b) (c) (d)

Teacher: So what did you do by putting that line in that has to do with Dahlia brought up? What are you doing with your line?

Shar: Making two more squares (as shown in e).



(e)

Teacher: But what's the important about what you said that she is doing? What word did you have to use there? What is she doing when she put the extra line in? What's she doing with the parts that you said was important?

Dahlia: She had equal parts?

Teacher: You're making them equal parts. That's the important thing you've done. And then what did you decide after you make all the parts equal? What are you saying about the brown?

Shar: There's one colored in and one fourth

Teacher: And so it's one-what's the whole? Can you show with your finger what the whole is? The whole rectangle? Does everyone agree that's the whole? Yes. Now, how many parts are cut into equal?

Shar: Huh?

Teacher: How many equal parts are cut into?

Shar: Two.

Teacher: Count all the parts in the whole rectangle.

Shar: One, two-four.

Teacher: Four. And then how many are shaded?

Shar: One.

Teacher: So, so you wrote?
Shar: One fourth.

The EML 2010 cohort already establishes the idea of "equal," so the teacher helps Shar fill in her incomplete explanation. The teacher made no intervention when Shar provided her initial explanation on the board at the beginning of the class, but the teacher prompts Shar further why she adds a line, what is the whole, and what kind of parts are because these are established knowledge in the class. In this process, Shar develops her explanation over time with the interaction with other students and with the teacher's support (see Table 4.4).

Table 4.4. The trajectory of Shar's explanation for the second problem

Instructional events	Type of explanation	Trajectory of Shar's explanation
Individual work	Written explanation	I know because I counted the brown one then every square.
Discussion about the first problem	Initial verbal explanation Repeated verbal explanation upon the teacher's request Details added by the teacher's support	That this (pointing to the right side) can be two squares here. I was looking that this (the left side) one was shaded in and there was- this rectangle (the right side) was two squares. So, it was one-fourth. I counted all these.
Discussion about Coretta's drawing	Initial verbal explanation	That one-fourth is the same about this one because all it is a split down there, but it's the same thing as that one, but it just has a split right there.
Macaulay's challenges of the new proposal (1/3) about the answer for the second problem	Initial verbal explanation Details added by the teacher's support	It's one-fourth because this is a rectangle and these are all squares and there's like these two rectangles and this-two squares in this one, and I-and then… (What did you do by putting that line in that has to do with Dahlia brought up?) Making two more squares. There's one colored in and one fourth (How many equal parts are cut into?) Two. (Count all the parts in the whole rectangle) One, two-four. (So you wrote?) one-fourth.

After hearing Shar's explanation, the teacher asks Shar to strip off the sticky line and gives a turn back to Macaulay to explain his proposal of 1/3 to the class. Before Macaulay gives his explanation, the teacher checks with the class whether they understand Shar's explanation.

Macaulay:	Well, the-the-but-you-there is only three shaded in. And I-no, not three shaded in, but three total sq- two total squares and one-one rectangle.
Teacher:	One whole. And how many squares are in it?
Macaulay:	Three.
Teacher:	Well, it's rectangle. One, two, three. And so therefore you are calling it?
Macaulay:	A one-third.

Macaulay stammered at the beginning but manages to explain that there are two squares and one rectangle at this time. Unlike the way in which the teacher follows up Shar, she does not ask Macaulay whether the parts are equal or not. After Macaulay's explanation, the teacher asks the students to decide whether the original second drawing is 1/3 or 1/4. Amani volunteers to explain.

Amani:	One-fourth.
Teacher:	Why do you think it's one-fourth?
Amani:	Because, even though it does not have a line, it can still be one-fourth.
Teacher:	Even though it doesn't have a line in it, it's still one-fourth. Can you say why?
Amani:	Because there is one piece shaded and if you were to put uhm, get that other rectangle and put a line through it, it would be one-fourth.
Teacher:	And what-why does putting the line into it make it clear that it's one-fourth? Amani (attaching the sticky line on the drawing)?
Amani:	Because it makes it look like there's four pieces instead of three.
Teacher:	And what kind of pieces?
Amani:	What?
Teacher:	What kind of pieces?
Amani:	Four pieces.
Teacher:	What kind of pieces, Dahlia?
Dahlia:	Equal.
Teacher:	Equal pieces. So when you put the line in, you're making all pieces equal. And that's what you say it's one-fourth. But, are you saying

	that even if you don't put the line in, we can still tell it's one out of four equal pieces? And how can you see that, Amani? Even if I don't use the line, how can you tell? Like, can you picture it even if I don't put the line up there?
Amani:	Uh, huh.
Teacher:	What are you doing in your mind?
Amani:	I'm thinking of it as-you can- I don't know.
Teacher:	What do you-what do you think Amani's doing? 'Cause she's saying she can decide [that it's one-fourth even if I don't stick the line-on there.]
Amani:	(whispering to herself) [I just know that it's one-fourth.]
Teacher:	What do you think she's doing? Michael?
Michael:	Um, I think she's making the parts equal in her mind.
Teacher:	Are you making the parts equal in your mind?
Amani:	Uh, huh.
Teacher:	Okay, does everyone understand what Amani's doing? Okay.
Students:	Yes.

In supporting the answer of 1/4, Amani's explanation has a quite similar structure that was discussed in the previous discussion: adding a line does not change the fraction. The teacher asks Amani to explain why it is still 1/4 if it does not have a line, but Amani explains how it would be 1/4 after putting a line. When the teacher asks a reason for the action (putting a line), Amani still focuses on the number of pieces after adding a line. The teacher then supports Amani's to use the established knowledge—Dahlia's idea about equal parts—in her explanation. Dahlia's idea about equal parts is already approved by the members of the classroom in the previous exchanges, so it serves as axiom to explain how to name a fraction. In other words, making equal parts is accepted without further justification, thus the students could use it to jump into the main argument. However, Amani could not provide further explanation how she could see 1/4 without adding a line. At this point, Amani does not have available mathematical resource to prove how she calls it 1/4 without adding a line. Michael provides his interpretation that Amani imagines adding a line in her mind.

Teacher:	So, what's the reason that many people, and Macaulay is one of them, and many people in this room, think it's one-third? What's the reason for that? Can I see some other people's hands I haven't seen yet today? Who else thinks they understand why? Even if you didn't think so or if you do. Why might somebody call that one-third? Okay, Jaclyn? What do you think?

Jaclyn:	Because they're not ima-imagining it equal.
Teacher:	So, what are they-what are they doing that gets them the three?
Jaclyn:	Uh.. Just looking at it.
Teacher:	And what do they see?
Jaclyn:	Three parts.
Teacher:	Three parts, right? Count them with me. One, two, three. And how many are shaded?
Jaclyn:	One.
Teacher:	So, that would lead somebody to say one-third.

After making Amani's idea clear, the teacher turns students' attention to understand the reasoning behind why somebody calls it 1/3. At this point, the teacher invites all students to understand the reasoning. Jaclyn uses Amani's idea about imagining and explains the reference of three. The teacher reminds the established knowledge that making equal parts is a key idea for naming a fraction and adding a line provides a tool to make parts equal. As indicated in the teacher's language use, "you don't really have to put the line," adding a line is a tool that makes it easy to see the equal parts, but is not a mandatory thing to do. After making it explicit, the teacher asks "why you can't call that one-third?" The use of modal verb of "can't" is interesting compared to the previous exchanges because it excludes the possibility of accepting 1/3 as an answer anymore. The teacher gives a turn to Macaulay to explain why it cannot be called 1/3, but Macaulay expresses that he does not want to explain it by shaking his head. The teacher emphasizes the importance of listening carefully and gives a turn to Eric to explain.

Eric.:	The rectangle isn't equal with the two squares. Like, it's not a square.
Teacher:	Can you say a little more clearly please?
Eric.:	The rectangle is supposed to be two squares, but it's just one.
Teacher:	Okay. This (pointing to the right side of the rectangle) has to be equal with?
Eric.:	The two other squares.
Teacher:	The two others. And this is much bigger. So, you don't have three. So, can someone say we don't have in this? Do we have three equal parts or not? Does anyone think these are three equal parts right here? Does anybody think that? Okay, we can't call that one-third, because the parts are not equal. What about this one (pointing to the first problem)? How come we could call this one one-third? Why was that find? Macaulay, can you explain why that one [was one-third?]
Macaulay:	[Because it's all equal.]

Teacher:	'Cause the parts are all equal. So, Macaulay, can you explain the difference between this drawing (pointing to the first problem) and this drawing (pointing to the second problem)? They both have a brown square, but why is this one (pointing to the first problem) one-third and this one (pointing to the second problem) is one-fourth? Can you explain that now?
Macaulay:	Because the one on the bottom (the second problem) can't be, is not equal with the…
Teacher:	But what about the top one (the first problem)?
Macaulay:	Top one (the first problem) is equal 'cause it has three equal groups.

Eric points out that the parts are not equal—one rectangle and two squares—and explains that the rectangle on the right side of the rectangle needs to be divided into two squares. The teacher makes a transition from focusing on the shape of each piece (rectangle is not equal with the two squares) to focusing on the size of each pieces (much bigger), examines whether the three parts in the second problem are equal, and concludes that it cannot be called 1/3. The teacher returns to the first problem and gives a turn to Macaulay, who provided a correct answer but incomplete explanation for the first problem and proposed the incorrect answer of 1/3 for the second problem, to re-explain the first problem, even though he did not raise his hand to explain. Macaulay initially missed the key idea of "equal" for the first problem, but now adds "equal" in his explanation for the first problem at this point. When Macaulay proposed the incorrect answer of 1/3 for the second problem, he made a reference to the agreed-on answer for the first problem—"the same as the top one"—so the teacher helps Macaulay figure out the difference between first drawing and the second drawing. Even in an incomplete form, Macaulay is able to see the difference between the first drawing and the second drawing. After Macaulay's explanation, the teacher asks other students to explain what Macaulay just said. Javonte, who also wrote the incorrect answer during individual work, has the opportunity to explain the difference between the first problem and the second problem.

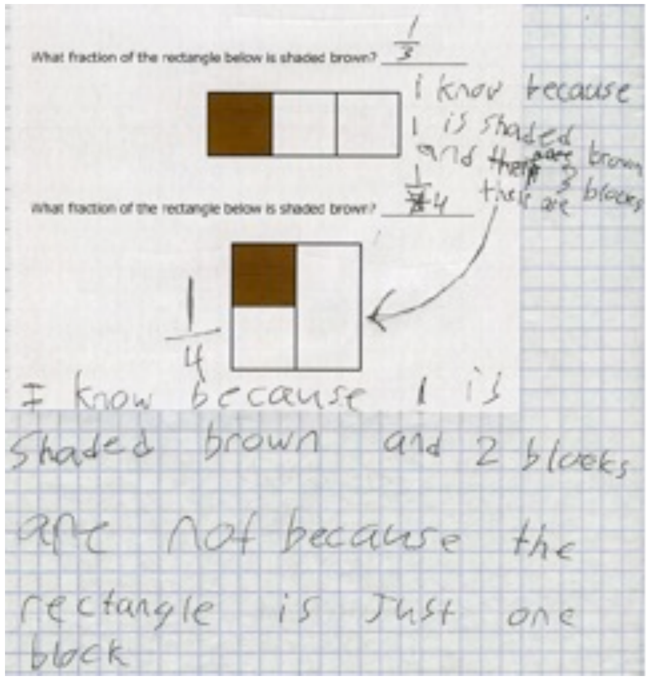


Figure 4.23. Javonte's written explanation in his notebook

Javonte:	Cause, um…
Teacher:	What fraction of the top one is shaded?
Javonte:	One-third.
Teacher:	One-third. And how come you can call that one-third?
Javonte:	Cause there's three parts.
Teacher:	What kind of parts?
Javonte:	Equal parts.
Teacher:	Equal parts. And how many are shaded in the three equal parts?
Javonte:	One.
Teacher:	One. So, it's called?
Javonte:	One-third.
Teacher:	All together, very good. It would be?
Javonte:	one-third.

After Javonte explains the answer of 1/3 for the first problem, the teacher shaded another piece on the first drawing to explain 2/3 and shaded one more piece to explain 3/3. The teacher then gives another opportunity to Macaulay to explain the second problem, who initially provides an incorrect answer of 1/3 for the second problem.

Teacher:	Okay, who can explain about the bottom one? Macaulay, do you wanna explain the bottom one?
Macaulay:	It is one-fourth, because I- because at my old school that- if it was in the whole entire thing it was- it was the number and my teacher

> said-also said that if it would-had three total squares and, and… had one big… rectangle. It should… be a one-fourth because there's three-three squares and one rectangle.

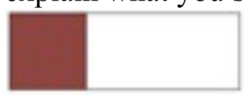
In the previous exchange, Macaulay got the clear idea that the first problem has three equal parts, but the second problem does not. At this time, he goes back to making a reference to his previous teacher and describes "three squares and one rectangle" with stammering a lot. It is not clear where "three squares and one rectangle" comes from, but it seems that he tries to match his explanation with the answer. Still being incomplete, but Macaulay changes his view that the answer for the second problem is 1/4. Through the engagement in a whole-group discussion, Macaulay develops his explanation over time with the interaction with other students and with the teacher's support (see Table 4.5).

Table 4.5. The trajectory of Macaulay's explanation

Instructional events	Type of explanation	Trajectory of Macaulay's explanation
Individual work	Written explanation for both problems	I know because 1 is shaded in and there are 3 squares.
Discussion for the first problem	Initial verbal explanation	I-I picked-and it says, what fraction of the rectangles is shaded brown? And I-and I- my answer was one third of the-[of the-]… I put one-third of the rectangle that's shaded brown.
	Repeated initial explanation upon the teacher's request	Well because, there is three squares and one of them is colored in. So you pick the one- you write the number that is like the… amount of that is shaded in, then you write the whole entire numbers without all the squares in it.
Discussion for the second problem	Initial explanation with the disagreement about the answer of ¼	Because it doesn't have a line through it, so that's just three.
	Repeated initial explanation	Well, the-the-but-you-there is only three shaded in. And I-no, not three shaded in, but three total- two total squares and one-one rectangle.
Comparison between the first problem and the second problem	Explanation about the answer for the first problem	Because it's all equal.
	Explanation about the answer for the second problem	Because the one on the bottom (the second problem) can't be, is not equal with the…
	Explanation about the answer for the first problem	Top one (the first problem) is equal 'cause it has three equal groups.
	Explanation about the answer for the second problem	It is one-fourth, because I- because at my old school that- if it was in the whole entire thing it was- it was the number and my teacher said-also said that if it would-had three total squares and, and.. had one big… rectangle. It should be a one-fourth because there's three-three squares and one rectangle.

The teacher then creates a private space for the students to re-check their original answer and to revise it if necessary. After a minute later, the teacher returns to Jaclyn's idea that was put aside.

Teahcer: Can someone describe this picture? I don't wanna hear the fraction. Just describe what you see. Your explanation should start with, "I see a rectangle", and then go on from there. Who can explain what you see up there? Okay, Karina, what do you see?



Karina: Um, I see the square that's shaded [and then a rec-]

Teacher: [A rectangle.] It's a rectangle. Yeah.

Karina: A rectangle that's halfway shaded in, but it can't be one-it can't be one-half.

Teacher: Because?

Karina: Because, they all gotta be equal parts.

Teacher: Okay, so, could you come up and then picture into one-half? I want you listen carefully to what Karina just said and then I'm gonna ask somebody to explain it after she makes the picture into what she thinks one-half.

Karina: Well you can.. (drawing the following drawing and then erase it)

Teacher: You want an eraser?

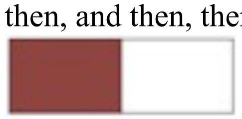
Karina: Oh.

Teacher: Sorry, I didn't mean to make you use your finger.

Karina: Can I erase the whole thing?

Teacher: Leave that 'cause I think it's confusing if you don't.

Karina: This is really one-half because one, one, one side is shaded in, and then, and then, there are two parts.



Teacher: There's a word missing in your explanation. The word that Dahlia reminded us of today. What part?

Karina: Equal. Because they're both equal.

Teacher: Okay, turn around and explain the drawing to the whole class now.

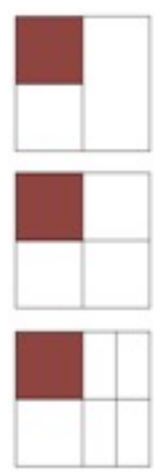
Karina: Um, there's only one part shaded in, so-so that has to be one-so that has to be one, and then there's two parts in all. So it's equal. And then, that has-that's how it's one-half. So, it can't be just right here (pointing the line in the original drawing), because it can't be even one-half, because it's not in the right-the line isn't in the right place.

Returning to Jaclyn's idea, the teacher first asks the students not to blurt out the answer but makes a space for them to describe the drawing. In refuting Macaulay's idea about 1/2, Karina provides her reasoning as refutation but complicates the meaning of "half " ("halfway shaded in" vs. "can't be one-half") in her explanation. As a reason for refuting 1/2 which Macaulay initially proposed, Karina explains that the drawing does not satisfy with the idea of "equal parts" for naming a fraction. To supplement Karina's explanation, the teacher invites Karina to the board and to draw 1/2. After drawing 1/2 with the same whole, Karina provides an explanation. After the teacher points out that something is missing in her explanation, Karina fills the idea of "equal" in her explanation. After Karina's explanation, the teacher opens up the discussion by asking for comments. Two students, Hala and Terrence, express their agreements with the answer.

Teacher:	Comments? So, when I say comments, I'm asking if you agree or disagree with what Karina just showed. Do you agree or disagree what she just showed us? Well, now I see more hands. So, you didn't know what I meant by comments. Hala, what do you think?
Hala:	I agree, because it's just like what she said, before it wasn't really one-half until you.. 'cause one-half is really in the middle things, I think. Not like a quarter of it. And that's why I think that it's one-half.
Teacher:	Okay, Terrence. Can you explain? Do you agree or disagree what Karina did?
Terrence:	Um… I kind of agree.
Teacher:	Because?
Terrence:	Because, like, the line between both of them with the shaded part, there's two boxes, and so the one that's shaded, it should be one-half. So.
Teacher:	And why do you think it's one-half? Can you finish what you are saying?
Terrence:	Because there's two boxes and [there's one that's shaded.]
Teacher:	And? Are they equal or not equal?
Terrence:	Equal.

Both for Karina's explanation and Terrence's explanation, the teacher directly address the incompleteness of the explanation so that they could fill up the missing idea in their explanations. The teacher then asks the students to draw the rectangle and represents 1/5 of the rectangle in their notebooks. In doing this, the teacher is able to extend making equal parts (three equal parts for the first problem and four equal parts for the second problem) and helps students to practice equal partitioning in their grid-notebooks. After this short activity to make five equal parts and to color one of them, the teacher wraps up a whole-group discussion about the brown rectangle problem on Day 2.

For the next few days, the teacher holds Coretta's idea for a whole-group discussion but does not have enough time to re-launch the discussion. In the first session of Day 6, the teacher revisits Coretta's diagram. Before re-launching the whole-group discussion about Coretta's idea, the class reviews the working definition of a fraction, which was developed through the discussion about the blue and green rectangle problem in Day 3. The working definition of fraction is posted on the left side of the board and three drawings are posted on the right side of the board. The first drawing is the original drawing for the second problem, the second drawing is Shar's drawing, and the third drawing is Coretta's drawing.

Working definition of a fraction • Identify the whole. • To identify a fraction, divide the whole into d equal parts. d is the denominator. • Write $\frac{1}{d}$ to show one of the equal parts. • If you have d of $\frac{1}{d}$, then you have the whole, or $\frac{d}{d}$. • If you have n of those equal parts, then you have $\frac{n}{d}$. n and d are whole numbers (in fifth grade). $d \neq 0$	

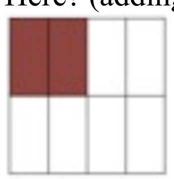
The teacher first asks a volunteer to explain why the first drawing cannot be 1/3. Shelly explains that "because there were three parts but they weren't equal or split up evenly." The teacher emphasizes that it is important to make equal parts to name a fraction and explains that many people draw the line to make equal parts by pointing to the second drawing on the board. The teacher asks "what fraction of the whole was the brown area" and Hala explains the it is 1/4. The teacher re-introduces Coretta's idea about adding additional line and asks the students to talk with a partner whether it is still 1/4 or not after adding the additional line as Coretta suggested. During a partner work, the teacher circulates the class to hear what they decide, including 2/8, 4/8, and 1/6. After a minute later, the teacher re-convenes the class to open a whole-group discussion. The teacher first surveys how many students think that it is still 1/4 even after adding the line, but only few students raise their hands. Confirming that most students think that it is not 1/4, the teacher asks someone to explain why he or she thinks that it is not 1/4, using the definition of fraction on the board. The teacher gives a turn to Ahmed.

Ahmed: The whole is the big square and-

Teacher: (tracing the whole with the marker) Okay, the whole is the big square.

Ahmed: I thought that the fraction would be, if you put a line through the other one, just like-

Teacher: Here? (adding another vertical line)



Ahmed: Uh, huh. I would think that it would be two-fourths.

Teacher: Okay, how did you get two-fourths?

Ahmed: Two out of eighths.

Teacher: Two-eighths? How did you get two-eighths?

Ahmed: Because if you count all of them, there's eight.

Teacher: Just a second. Could you hear Ahmed okay? Okay.

Ahmed: And there's two shaded in.

Teacher: Okay, so how many equal parts are there?

Ahmed: Eight.

Teacher: One, two, three, four, five, six, seven, eight. Let's count by eighth. Can you count by eighth with me? One eighth,

Students: Two eighths, three eighths, four eighths, five eighths, six eighths, seven eighths, eight eighths.

Teacher: How many is Ahmed saying is shaded-are shaded?

Students: Two.

Teacher: Two. So, you are saying two-eighths?

Ahmed: Uh-huh.

Reminding the established knowledge that the class collectively constructed—the working definition of fraction—, the teacher gives an opportunity for students to propose why they think that it is not 1/4. Ahmed, adding another line, explains how he gets 2/8. After Ahmed's explanation, the teacher asks Ahmed how many equal parts and then asks the students to count by 1/8. The teacher asks the students to raise their hands if they think that it is 2/8. As most of the students raise their hands, the teacher looks for anyone who disagrees with Ahmed. Coretta shows her disagreement with Ahmed because there are two answers.

Teacher:	Okay, so lots of people agree with Ahmed. Does anyone disagree with Ahmed? Coretta, you disagree?
Coretta:	It's actually two answers.
Teacher:	Okay.
Coretta:	You can actually take the line away or you can actually keep the line.
Teacher:	Okay, you can take the line away or you can keep the line?
Coretta:	Right.
Teacher:	So, he kept the line. How would you take the line away? Here, I'll- why don't we get a clean drawing? Could you come up and say what you mean by taking the line away?
Coretta:	(coming to the board and explaining) There's two answers to it.
Teacher:	Okay.
Coretta:	You can actually take this line away, and you can get four squares back. And then you can actually add a line. So, there's two answers to it. You can have either two-eighths or one-fourth, [if you are taking this line away.]
Teacher:	[Okay, so you] so you're saying that if you take the line away, then you can say the pictures shows one-fourth?
Coretta:	Right. But, if you add the line to this side, you have two-eighths. So, there could be one answer or two.
Teacher:	So, how could we show taking away the line? Do you want me to just sort of-
Coretta:	This one.

Coretta's disagreement does not aim to reject Ahmed's answer of 2/8. Rather, Coretta proposes the existence of multiple answers, depending on the treatment of the line—if taking the line away, it is 1/4, whereas if keeping the line, it is 2/8. Comparing to Coretta's previous claim about two answers in Day 2—either it is 1/4 or not 1/4 (1/6) which are mutually exclusive ones—, she advances the idea of making equal parts into making equivalent fractions. After checking whether other students understand Coretta's explanation, the teacher invites students to comment on Coretta's explanation.

Shelly: Well, if you think of it like this, um… If you do what Ahmed did and added another line, you'd have two-eighths. But, what if like you would get three answers, if you do two more lines across that way, and then if you just kept adding lines, you'd get different answers. [You could get like-]

Coretta: [When you say-when you say-] do you mean this way (gesturing the horizontal), or this way (gesturing the vertical)?

Shelly: I mean like… If

Teacher: Here. Do you wanna come up and say what you're doing?

Shelly: Yeah.

Teacher: So Shelly, you're ask-you're suggesting we can make more lines?

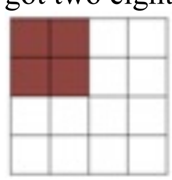
Shelly: Uh-huh.

Teacher: Let her try. Let her do it.

Coretta: So, you are saying, if you add another line-

Teacher: Let Shelly do it what she's showing.

Shelly: Okay, so if you add another line, like, what we already did and you got two eighths, uh, well, you can add a line here, and a line here.



Teacher: Now, how many equal parts have you made, Shelly?

Shelly: Sixteen.

Teacher: Sixteen?

Shelly: Yeah, 'cause you get-

Teacher: And how many are shaded?

Shelly: Four sixteenths.

Teacher: Can you write four-sixteenths to the left of the pictures? Nice and big. Coretta..

Shelly: And

Teacher: Go ahead, Shelly.

Shelly: And then if you kept splitting them up evenly and add more lines or something like that, you would just get a different answer every time.

Coretta: So, it's kind of like the multiplication problem that we're already doing or that we already did. The fives, where you kept on going down and down, you could get more answers by splitting it up evenly.

Shelly: Yeah. And you don't just have two answers or anything. It just pretty much.. yeah.. it's- that one- it can be any answer pretty much.

Coretta: If you keep- I see what you're saying. If you keep on splitting it up evenly, like how it's started out this way (pointing to the first drawing), and that's not even, or equal. You have to split it up (pointing to the second drawing) into another equal piece. And if you don't want that equal piece, you can split it up to two-eighths (pointing to the third drawing), two eighths can give you one fourth (pointing to the fourth drawing), or it can give you two-eighths, either way, these two (pointing to the third and fourth drawing) can be equal, if you multiplied, or either way. And then this one (pointing to the fifth drawing), if you split this one up again (pointing to the fourth drawing), you would get four-sixteenths.

Shelly's comment on Coretta has mathematically similar structure with Coretta's disagreement with Ahmed in Day 6, as well as with Jaclyn's comment on Macaulay and Coretta's comment on Shar in Day 2. Applying "what if" question, Shelly keeps adding more lines and reaches at three different answers. Coretta makes a connection between adding more lines geometrically and multiplying numerators and denominators with the same number algebraically to make equivalent fractions. The teacher then gives turns to Bernard and Thailee to provide comments.

Bernard: Well, if you keep adding lines that are always equals, you can always get a way back. Because if you simplify four-sixteenths, you can get one-fourth.
Teacher: Okay, so you can always get a way back by taking lines back out?
Bernard: Yeah.

Extending the idea that Coretta makes a connection between a geometric method and an algebraic method, Bernard makes a connection between removing the lines geometrically and simplifying the fraction algebraically. The teacher does not scale up the idea of simplifying the fraction, but repeats how to get back to 1/4 by removing the lines in Shelly's drawing. In the meanwhile, Thailee raises her hand to express that she does not follow up the conversation between Shelly and Coretta.

Teacher: You don't get anything what they're saying? Okay. So Thailee is asking what Coretta and Shelly are showing. Could someone explain what happens from this picture (pointing to the second drawing) to this picture (pointing to the third drawing)? And pay

	attention, Thailee. Okay? Could someone explain how do we get from one-fourth to the two-eights picture? Michael?
Michael:	Um, to make it easier, all you have to do is just double, two times- or one times two, and that'd be two. And then four times two is eight.
Teacher:	How do you see that in the picture, Michael?
Michael:	Um.. because.. because… um.. if you divide it down, like both ways, and then it would be um, two would be shaded and there's eight equal parts.
Teacher:	Uh-huh.
Michael:	So that would make it two-eighths.

Michael attempts to explain numerically, by multiplying two both to the numerator and the denominator, but the teacher asks him to use the drawing for explaining. After the teacher repeats Michael's explanation, she checks whether Thailee understands how to get 2/8. The teacher then asks someone to explain what Shelly did to get 4/16. Kassandra takes a turn to explain.

Kassandra:	She divided eighths into two, into two, and, eight plus eight equals sixteen.
Teacher:	And how many of sixteenths were shaded?
Kassandra:	Four.
Teacher:	Four. Okay, let's count by sixteenths in the picture. Can you do it with me? One sixteenth, um, Qayshawn? Ready? One sixteenth,
Students:	Two sixteenths, three sixteenths, four sixteenths, five sixteenths, six sixteenths, seven sixteenths, eight sixteenths, nine sixteenths, ten sixteenths, eleven sixteenths, twelve sixteenths, thirteen sixteenths, fourteen sixteenths, fifteen sixteenths, sixteen sixteenths.
Teacher:	When you get to sixteen-sixteenths, what have you get?
Students:	One whole.

After Kassandra's explanation, the teacher suggests the students to count by 1/16 and elicits the concept that 16/16 equals to one whole. The teacher then asks whether the brown is the same amount of the whole. Ahmed responds that the brown is the same size of the whole and the teacher further asks him to prove it.

Teacher:	Is there any way you could prove that?
Ahmed:	Cause like, if you have them like in any way like, like, four, four mini squares or a line through it, it's still same size brown.

Teacher: Okay, do you think you use this (showing the sticky brown square) to come and show it? Here's a sticky one. Could you use what we did the last week? Could you prove that the brown is the same size? Are the wholes the same size? Is this whole, this whole, this whole, and this whole the same size? Yes? Okay, so now Ahmed 's gonna show that the part is the same size also. Can you take that from one to the next and show us how it's the same each time?

Ahmed: Like that (overlapping the sticky brown square into the brown part of four drawings)

Teacher: Okay, so tell them what you figured out. Turn to the class. Can you turn your face to the class, please?

Ahmed: I figured out that if you put this square (sticky brown square) to each of these squares, it's still the same size. And even if you put a line through it, it's just still the same size.

After Ahmed's explanation, the teacher checks with Thailee whether she understands Ahmed's explanation. The teacher comments that there are different ways to name a fraction and asks another name for 1/4. Kassandra answers one-quarter, Coretta answers two-eighths, and Zahara answers four-sixteenths. Shelly expresses that she has another way of naming a fraction and explains how one-half, two-fourths, three sixths, and four-eighths are equal. The teacher lastly turns back to Coretta and what she decides after putting the line.

Coretta: I decided you can give more answers, that you are actually seeing and there's much more than that. You can get, if you go over four-sixteenths, you can make it smaller. You can get the hundredths.

Teacher: Kind like what?

Coretta: Hundredths or notebook paper. You got so small, you got a lot of fractions. If I split it, down the middle, and, or this is called one region, you will get how many numbers or squares in the page must be equal in the shaded piece.

As the beginning of Day 2, the second problem is introduced as the unequally partitioned rectangle. To name a fraction for the unequally partitioned rectangle, Shar adds a line to make four equal parts. Coretta came up with the same answer with Shar for the second problem, but provided a comment to Shar about what if adding an additional line in Shar's drawing, which results in the unequally partitioned rectangle again on Day 2. When Coretta provided a comment to Shar, she had mutually exclusive mathematical ideas: the answer might be either 1/4 or not 1/4 (1/6). Through the engagement in debating whether it is still 1/4 or not and in hearing Ahmed's idea that adding an additional line to make equal parts results in 2/8 on Day 6, Coretta challenges Ahmed that 1/4 and 2/8 are not mutually exclusive one, but compatible one by adding or removing the line. Coretta's conjecture that there are two answers for her suggestion (1/4 and 2/8) is further challenged by Shelly that adding more lines produce different, but equivalent, fractions. After Shelly's challenge, Coretta acknowledges that there are more than three answers, but extends the idea that the unit gets smaller by adding more lines for equivalent fractions. The EML 2010 cohort develops the idea that adding a line is a key idea for naming a fraction because it makes equal parts and extends this concept to equivalent fractions, even though the term of "equivalent fraction" is not formally introduced and the algebraic method is not further taken up.

Summary

The EML 2010 students initially propose one answer (1/3) for the original drawing of the first part of the brown rectangle problem and initially propose two answers (1/4 and 1/3) for the original drawing of the second part of the brown rectangle problem. In explaining the brown rectangle problem on Day 2, the EML 2010 students initially miss the key idea of "equal" for naming a fraction; decline the invitation to the board for explaining (e.g., Jaclyn, Shar), use the ambiguous and inaccurate language (e.g., "what if there was no picture," "halfway," "middle thing"); uses demonstrative pronouns (e.g., "this"); ground the rationale on non-mathematical reasons (e.g., "my teacher said"; "learned from my fourth grade class"); and describe an action taken (e.g., "I counted") without specifying the object.

For the first part of the brown rectangle problem, Macaulay proposes the correct answer (1/3) but provides the incomplete explanation (i.e., missing "equal"). Macaulay's explanation is challenged by Jaclyn who has a record of the same answer with Macaulay in her notebook but makes a comment about deleting the existing line to make the original drawing of the first part of the brown rectangle problem unequally partitioned. For the second part of the brown rectangle problem, Shar proposes the correct answer (1/4) but provides the incomplete (i.e., missing "equal") and partial explanation ("two squares"). Shar's explanation is challenged by Coretta who has a record of the same answer with Shar in her notebook but makes a comment about adding an additional line to make Shar's drawing unequally partitioned. The EML 2010 students challenge the correct answer by asking "what if" question which makes the equally partitioned rectangle to the unequally partitioned rectangle and makes the unequally partitioned rectangle to the equally partitioned rectangle by adding or deleting a line. Throughout the discussion, the EML 2010 students review different claims about the second part of the brown rectangle problem and expand the idea to equivalent fractions on Day 6. For example, Coretta argues that the answer is either 1/4 or not 1/4 (1/6) on Day 2, but she revises her claim that the answer is either 1/4 or 1/8 after hearing Ahmed's explanation on Day 6. This claim is expanded by Shelly that the answers are 1/4, 2/8, and 4/16 and then generalized by Coretta that there are many answers.

To support students' development of mathematical explanation for the brown rectangle problem, the teacher distributes an equal opportunity to explain rather than heavily controlling the sequence of proposals; serves as a delegate to make the idea to be fully addressed in a public space; gives a turn back to the students who initially proposed the incorrect answer and involves them in constructing a mathematical explanation; invites students to the board so that the students supplement their verbal explanation with pictorial representation; delays the evaluation of the answer and the rigorous inspection of the completeness of explanation at the beginning but addresses the incompleteness of the explanation after developing the key ideas; supplies sticky lines and cutouts to support the explanation; and increases the level of mathematical supports over time (no substantive mathematical supports, including clarifying what the rectangle refers to and remediating errors, during set-up stage and the individual work). However, the teacher neither accepts the non-mathematically grounded reasoning nor advances into the idea of making equivalent fractions algebraically that is not collectively established by the EML 2010 cohort.

4.6. The Case of EML 2013

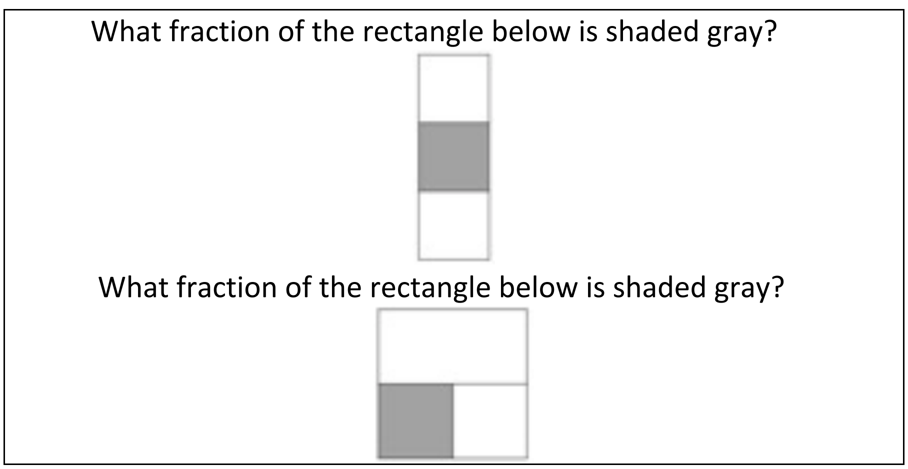
Preview

The brown rectangle problem is introduced on the first day of the EML 2013. The teacher posts two problems together (the equally partitioned rectangle and the unequally partitioned rectangle) on the board and asks the students to write down the answer in their notebooks. The teacher does not clarify what "the rectangle" refers to during launching the problem and does not provide substantial mathematical supports during individual work. For the first problem, two answers are proposed by Liberty and Deshawn. The first proposal of 1/3 is made by Liberty. After her initial explanation, the teacher asks other students to repeat, add on, and agree or disagree with Liberty's explanation. Through this process, Liberty's initial explanation is elaborated by Deshawn's repeat, Tina's add-on, Renee's agreement, and April's repeat. In responding to the teacher's attempt to elicit another answer, Deshawn makes the second proposal of 1/2. After Mark repeats Deshawn's explanation, the teacher asks other students whether they agree or disagree with Deshawn's answer. As Kallie expresses her disagreement, Deshawn changes his mind from 1/2 to 1/3.

For the second problem, three answers are proposed by Kadeem, Anaya, and Kallie, respectively. The first proposal is made by Kadeem. He explains that it is not a fraction without equal parts. Expressing the disagreement with Kadeem, Anaya proposes 1/4 by adding a line on the top of the rectangle. Expressing the disagreement with Anaya, Kallie proposes 1 and 1/2. The teacher wraps up a whole-group discussion on Day 1 because of the time constraint and then revisits the brown rectangle problem on the next day. On Day 2, Kallie changes her mind from 1 and 1/2 to 1/4. In reviewing three proposals, the EML 2013 cohort collectively develops three important ideas for naming a fraction: (1) what is the whole; (2) divide the whole into equal parts; and (3) name one of equal parts. The extensive detailed analysis of 20-minute of instructional interactions on Day 1 and 20-mintue of instructional interactions on Day 2 for teaching the brown rectangle problem in the EML 2013 is provided below.

Extensive Detailed Analysis

In the EML 2013, the brown rectangle problem is introduced on the second session of Day 1. In launching the brown rectangle problem, the teacher asks someone who does not have a turn to talk yet and then asks Tashawnah to read the agenda for the brown rectangle problem written on the poster: “Identify the fraction of area and explain answers about fraction names.” After a brief introduction about setting up the notebook, the teacher distributes a copy of the problem to students. Like the EML 2010, the teacher posts two problems (the equally partitioned rectangle and the unequally partitioned rectangle) together, in which each diagram has its own, but the same, written problem statement. In the EML 2013, the teacher uses the different layout of the brown rectangle problem, by rotating each rectangle to the 90 degrees counterclockwise than the drawing used in the previous years and shaded the middle piece rather than the left side of the rectangle in the first drawing, as well as the color of the shaded part. Like the previous years, the teacher does not clarify what “the rectangle” refers to at this point.



During individual work, the teacher circulates the classroom to monitor what the students write in their notebooks, but does not explicitly asks them to write a reason for the answer. All but one student do not request the teacher to clarify the problem statement and the teacher does not voluntarily provide substantial mathematical supports during individual work. After the teacher checks Deshawn’s answer and moves her step toward Tashawnah, Tashawnah asks for help to the teacher but the teacher responds that the class will talk about the problem soon and moves her step to the next student.

While the teacher circulates the classroom, she shares her observation that she saw different answers. In the previous years, the teacher announced that most of the students got the same answer for the first problem but they got the different answers for the second problem, but in the EML 2013, the teacher does not specify whether those different answers are observed for the first problem or the second problem or both problems. For the first problem, 23 out of 27 students wrote 1/3 in their notebooks. Two students, Deshawn and Anaya , overlapped two answers, 1/2 and 1/3, in their notebooks. Because two other students, Tenisha and Bria, completely crossed out their initial answers, it is not easy to discern which answer they came up with for the first problem before a whole-group discussion.

After three-minute of individual work, the teacher convenes the class to launch a whole-group discussion. The teacher again shares her observation that they have different answers, without specifying for which problem, and addresses that it is important to listen very carefully to others' ideas. The teacher then looks for a volunteer to give an explanation on the board. Inviting Liberty to the board, the teacher sets the expectations both for the explainer (speaking nice and loud) and the audience (listening carefully to Liberty's explanation and trying to understanding her explanation rather than agreeing or disagreeing with her). Upon the teacher's request, Liberty first reads the problem statement and provides an explanation.

> What fraction of the rectangle below is shaded gray? Well, I think that it's one-third because you have to count all the squares first, because the frac-the denominator is, is, what is uhm, is the, is the amount that you have, and the fraction is what they tell you to make the fraction like the fraction is what shaded gray. And I think that is one-third.

Liberty attempts to make a connection between the total number of parts in a whole (all the squares) and the component of fraction (denominator), but ends up with providing general description about terms (the denominator is the amount that you have and the fraction is what is shaded gray). Her explanation is characterized as: (1) not building a connection between her answer and her explanation; (2) partially making use of pre-defined mathematical term (i.e., denominator); (3) being confused between "fraction" and "numerator"; (4) mathematically incomplete (no indication about equal); (5) using geometric names (i.e., "squares"); and (6) using a modal verb (i.e., "have to").

After Liberty provides an explanation, the teacher asks her to write down the fraction on the board and then asks someone to explain what she said. In requesting for repeating, the teacher asks the students not to agree or disagree with Liberty, but to understand what she said. When Deshawn repeats "Uhm, the denominator was three, and the shaded-," the teacher pauses his explanation to check with Liberty whether she explained how the denominator was three. In the initial explanation, Liberty mentioned that the denominator comes from counting all the squares but did not specify that the denominator is three. Hearing very attentively what is initially offered by Liberty and what Deshawn additionally adds, the teacher gives a turn back to Liberty to specify how she knows that the denominator is three. Upon the teacher's request, Liberty adds a supplementary explanation, "Uhm, I knew that the denominator is three because there are three whole squares." Liberty termed "all the squares" in her initial explanation, but replaces it to "three whole squares" in her repeated explanation. Her use of the term "whole" does not match with the accepted mathematical definition of "whole" for naming a fraction. The teacher then checks with Deshawn whether he could hear Liberty's explanation and ask Deshawn whether there is anything else that Liberty said. Deshawn explains, "Uhm, she said that uhm, the denominator, the denominator was three whole squares." Using the same term, Deshawn perfectly captures what Liberty said. After Deshawn repeats, the teacher asks the students whether anyone has anything to add on Liberty's explanation. Tina points out that Liberty forgot to explain that the numerator is one. The teacher asks why the numerator is one and Tina explains "Because it is how much is shaded in." The teacher returns to the initial explainer, Liberty, and asks her whether she agrees with Tina's comment.

Through these exchanges, the initial explanation provided by Liberty is expanded through Deshawn's repeat, Tina's comment, and Liberty's confirmation. The teacher launches a whole-group discussion by asking not to agree or disagree with Liberty's explanation at the beginning, but checks whether anyone agrees or disagrees with Liberty's answer at this point. Renee raises her hand to express her agreement with Liberty and explains her reason for the agreement.

Renee:	Because one shaded and three is-two is not, and the total they have is three.
Teacher:	Could you say that again, Renee?
Renee:	Okay. So, there's three cubes and one shaded, so…
Teacher:	So?
Renee:	So, it's one-third-so one shaded so that's numerator and there's three that's denominator because that's how many there are.
Teacher:	Okay. Good. Can someone say what Renee just said? Who was listening carefully to Renee and can say what she said?

Before Liberty comes to the board to give an explanation, the teacher asks the students neither agree nor disagree. After Liberty provided an explanation for her correct answer and her initial explanation was expanded through Deshawn's repeat and Tina's add-on, the teacher asks the students whether they agree or disagree with Liberty's answer. Most of the students came up with the same answer with Liberty, so checking the agreement or disagreement functions as reaching an agreement about Liberty's answer and creating another opportunity to repeat Liberty's explanation. In responding to the teacher's request to explain the reason for agreement, Renee adds the information about unshaded parts but counts just numbers without specifying the objects (i.e., "three") at her first attempt, names "three cubes" at her second attempt, and makes a reference to the numerator and the denominator at her third attempt. The teacher asks someone who can repeat Renee's explanation again, but no one volunteers to repeat Renee's explanation. The teacher takes this as an opportunity to set the norm for both the explainer (speaking loud enough) and the audience (listen carefully). Renee repeats her explanation again, as well as consolidating the tacitly agreed-on explanation for the first problem.

> Okay. There's three squares and one shaded, so that's numerator on the top, and there's a total of three squares, that's a denominator.

In the repeated explanation, Renee changed the word "three cubes" to "three squares" and keeps the structure of initial explanation provided by Liberty. The teacher then asks someone to repeat Renee's explanation. April repeats:

> Yeah. So she had three cubes all together, so that's the denominator, and there's one cube shaded, so that's one-third.

April replaces the word "three squares" to "three cubes" again but makes explicit that it corresponds with the denominator. After April's repeat, the teacher continues to call for different proposals. In her third calls, Deshawn raises his hand to share his answer of 1/2.

Teacher:	Does anybody have different answer for the first question? For this question, does anybody have something, not, other than one-third? So, everybody got one-third for the first answer? Deshawn, you didn't get one-third?
Deshawn:	Mm-hmm. One-half.
Teacher:	For this one? (pointing to the first problem)
Deshawn:	Yes.
Teacher:	So can you explain how you got one-half?
Deshawn:	So, when I looked at the cube, I saw that there was one and there was two not grayed, that was gray, so I thought that it is one-half.
Teacher:	Okay, so can you come up and just point what you are saying? So, watch carefully what Deshawn is saying and see what you think about this.
Deshawn:	So, I thought that there's one shaded gray and these two (pointing to the unshaded squares) weren't, so I got one-half.
Teacher:	Can someone explain how we got one-half? Who listen carefully to explain how he got one-half? Mark, what does he say?
Mark:	He said that there's a two non-shaded cubes and there's one shaded. So, he-
Teacher:	-Okay, how did he get one-half?
Mark:	Uhm, by counting the two squares that are not shaded and one shaded.
Teacher:	So, two for you is not shaded ones and one was the shaded one. So you wrote one two?

Deshawn repeated Liberty's explanation of 1/3 in the previous exchanges, but makes a proposal of 1/2 to the class. Taking over the term "cube" that Renee and April used, Deshawn shares his reasoning but does not provide details what two unshaded gray corresponds and what one shaded gray corresponds in the form of fraction. Instead of asking agreement or disagreement right away, the teacher first invites Deshawn to the board to point out the pieces that he refers to and then checks whether other students understand Deshawn's explanation. In repeating Deshawn's explanation, Mark replaces the term "cubes" to "squares" but captures Deshawn's reasoning well.

The teacher first checks whether everybody understands Deshawn's explanation and then asks other students whether they agree or disagree with 1/2. When the incorrect answers for the second problem were proposed in the previous years, the teacher asked the students not to agree or disagree with the answer, but to understand the reasoning behind the answer. On the other hand, the teacher explicitly asks the students whether they agree or disagree with the incorrect answer for the first problem that Deshawn proposed. One reasoning might be that protecting the incorrect answers for the second problem, such as 1/3 or 1/2, plays an important role in developing the complete mathematical explanation for the brown rectangle problem, but protecting the incorrect answer of 1/2 for the first problem does not contribute to the development of mathematical explanation for the brown rectangle problem. Moreover, Deshawn's incorrect answer of 1/2 violates a part-whole relationship in the area model of fraction. In responding to the teacher's request, Kallie expresses her disagreement about the answer that Deshawn proposed.

Kallie:	Because I know his thinking. He's putting those two unshaded rectangles for the denominator and the one as numerator. So, I understand that it isn't correct-
Teacher:	Let's stop for a minute.
Teacher:	(To Deshawn) Is it your thinking?
Deshawn:	Yeah.
Teacher:	(to Deshawn) Does she understand your thinking?
Deshawn:	(nodding his head) Yeah.
Teacher:	(To Kallie) Okay, do you want to continue?
Kallie:	It's-what is supposed to do is, you're supposed to take, there're three squares and you're supposed to take the one that shaded and use that one as numerator, the-the-
Teacher:	Numerator
Kallie:	Numerator and the, number three, three squares as denominator.

Kallie's comment starts with expressing her understanding of Deshawn's reasoning and adds additional information about what corresponds to the numerator and what corresponds to the denominator. At the moment that Kallie points out that Deshawn's answer is incorrect, the teacher pauses Kallie's explanation and checks with Deshawn to see whether Kallie captures his reasoning. After Deshawn confirms, the teacher gives a turn back to Kallie to continue her explanation. After Kallie's explanation, the teacher asks Deshawn to comment on Kallie's idea and asks how he thinks about two answers.

Teacher:	Which answer do you-you think still it's one-half or do you think it's one-third?

Deshawn: One-third.
Teacher: Do you? Do you change your mind?
Deshawn: Yeah.
Teacher: Why did you change your mind?
Deshawn: Because I thought that the total was three and there's one shaded gray.

Kallie's disagreement neither serves as the proclamation of rejecting Deshawn's incorrect answer nor dampens Deshawn's courage to share a different answer. Instead, Kallie's disagreement makes it clear that Deshawn considers the denominator as unshaded parts, but the fraction in an area model is a part-whole relationship rather than a part-part relationship. After these exchanges, the teacher provides comments about how Deshawn and Kallie engage in the mathematical work:

- Deshawn: sharing a different answer, explaining his thinking, and changing his mind after listening carefully to the reasoning
- Kallie: understanding other's thinking and not just disagreeing right away.

The teacher then summarizes the agreement about the answer for the first problem and provides a complete explanation:

> I think, now we agree that the way to name this fraction count three equal parts and one of them shaded, so it's one is gray out of three equal parts, one-third.

After announcing that the class does not have enough time to fully engage in the discussion for the second problem, the teacher begins to elicit answers for the second problem. For the second problem, 12 out of 29 students (April, Madeline, David, Anaya, Liberty, Ty, Tashawnah, Kalvin, Renee, Jarvaise, Elysa, and D'lon) neatly wrote 1/4 in their notebooks. As Tashawnah expresses her struggles for understanding the problem to the teacher during individual work, there is a possibility that Tashawnah wrote the answer of 1/4 during a whole-group discussion, not during an individual work. Because several students completely crossed out their original answers and re-wrote 1/4 next to it, it is difficult to figure out all of the different answers that students came up with during individual work. However, most of the remaining 17 students recorded 1/3, one student (Deshawn) wrote 1/2, and two students (Kallie and Mark) wrote 1 and 1/2 in their notebooks.

Because of the camera angle, it is difficult to name all of the students who raise their hands to explain, but Kalvin, Liberty, and Aryanna raise their hands to explain. The volunteers are mixed of the student who wrote the correct answer (e.g., Kalvin, Liberty) and who wrote incorrect answers (e.g., Aryanna) in their notebooks. Kadeem gets a turn and comes up to the board to explain his answer.

Kadeem:	I think that it's not a fraction because all of the parts are not equally same shape.
Teacher:	Can you say one more time to the class?
Kadeem:	I think it's not a fraction because all the parts are not equal-equally same.
Teacher:	Can someone repeat what Kadeem said? Very nice, Kadeem. What did he say? Aryanna?
Aryanna:	Uhm, he said that he doesn't think it's a fraction because not all the parts are equal.

Even though Kadeem does not propose the correct answer of 1/4 for the second problem, he proffers the key idea for naming a fraction (i.e., "equal") that the teacher aims to elicit during the lesson. In his initial explanation, Kadeem adds an important word "equal" but does not use the word accurately as the teacher strives for—in naming a fraction, equal size matters than equal shape. Through repetition, the word "not equally same shape" translates into "not equally same" and then again into "not all the parts are equal." After Kadeem's initial explanation, the teacher does not seek for agreement or disagreement about the answer, but asks Kadeem to repeat first and then asks other students to repeat the explanation. Aryanna, who produced the incorrect answer of 1/3 by not considering the equal parts, gets a turn to repeat Kadeem's explanation. After repeating, the teacher opens the conversation for commenting, agreeing or disagreeing.

Kalvin, Renee, Liberty, Aryanna, Tina, Kallie, and Ahmed raise their hands to express that they want to give comments. The teacher gives a turn to Anaya, who passively raises her hand, to give a comment. After a few seconds of silence, Anaya expresses her disagreement and explains her answer of 1/4. The teacher invites Anaya to the board to show her thinking.

> I think it's one-fourth because, like he said, all of the fractions aren't the same, but you can make them same by dividing the line down the middle (gesturing to draw the line on the top piece of the rectangle).

Kadeem proposes the incorrect answer, but provides a useful tool for Anaya to use in explaining the reason for adding the line. Anaya does not build a correspondence between her answer and her verbal explanation, but addresses that the parts need to be the same and what mathematical work needs to be done to make them equal. Her language use of "fraction" is not accurate, but makes a reference to Kadeem's idea. When Anaya is about to draw the line on the poster, the teacher hands over a sticky line to Anaya. After attaching the line, Anaya provides her explanation again:

> Uhm, I divided it down the middle because, since it's not equal, you ***<u>have to</u>*** make equal.

In repeating her explanation on the board, Anaya changes her voice from "you can make them same" to "you have to make equal." For the shared understanding, the teacher requests other students to repeat Anaya's explanation by pointing out the idea emerged both in Kadeem's explanation and Anaya's explanation. The teacher says:

> Okay, can someone repeat what Anaya said? What she did and what she said? It actually goes very nice with what you said, Kadeem. Because Kadeem noticed that the parts weren't equal and what Anaya is doing has to do is with equal parts. So, these two answers go together very nicely. Can someone repeat what she just said?

The teacher gives a turn to VirShwan to repeat, but he does not provide an explanation. The teacher then asks Anaya to repeat her explanation:

> Uhm, you have to make the fraction the same since it's not equal. And you have to divide it by the part that wouldn't be equal.

In the repeated explanation, Anaya strengthens her voice by using a stronger modal verb of "have to." The teacher once again gives an opportunity for D'lon to repeat but he is not able to repeat Anaya's explanation for the second trial. Getting a promise from VirShwan to follow others' explanation, the teacher turns to Jarvaise to repeat Anaya's explanation.

Jarvaise:	She, she added like, she added a line to make it two more.
Teacher:	To make what?
Jarvaise:	To make like this whole piece to more pieces? (showing his notebook)
Teacher:	Uh-huh.
Jarvaise:	So I can actually-
Teacher:	So why did she do that?
Jarvaise:	(silent)
Teacher:	Can you say why did you add the line?
Anaya:	To make equal.
Teacher:	Can you say that again, Jarvaise? Why she added a line?
Jarvaise:	So I can make four (showing his notebook).
Teacher:	To make-
Jarvaise:	One-
Teacher:	What word did Anaya just say that you need to say? She said she didn't just make four parts, but she made what kinds of parts?
Jarvaise:	(shrug his shoulder)
Teacher:	(To Anaya) Say that again?
Anaya:	Equal parts.
Teacher:	Equal parts. The same size. Do you see that Jarvaise?
Jarvaise:	Yeah.

Anaya explains the reason for adding a line—to make equal—but Jarvaise's repetition conveys the product of adding a line—making two more pieces. Jarvaise sticks to the number of pieces after adding a line, even when the teacher asks him to explain the reason for adding a line. The teacher then gives a turn to Anaya to explain the reason of adding a line. When the teacher tells the answer that Anaya proposed, Kallie, who showed disagreement with Deshawn's incorrect answer for the first problem, raises her hand to show her disagreement with Anaya's correct answer for the second problem.

Kallie:	I disagree because uhm, the question was asking what fraction of the rectangle is shaded gray already before you add a line, and my answer for it was uhm, one and one-half.
Teacher:	Ah-ha.
Kallie:	Because you might say it's uhm, three, one-third, but he's right (pointing to Kadeem), it's not equal, so uhm-
Teacher:	That was Kadeem. Kadeem, is it right? okay.
Kallie:	Kadeem said it's not equal, so uhm, there's one-half below like a full rectangle.
Teacher:	Here, we can take the line off for a minute (taking off the sticky line). Anaya, I think that you can sit down. I think you did a nice job with explaining. I can take this line back off. What are you saying now?

Kallie:	There's uhm, the parts that are shaded would have to be one-half because below there is uhm, uhm, it's cut up into a half-
Teacher:	This right here?
Kallie:	And one above it, so that's how I got my answer, one and one-half.

Kallie expressed her disagreement with Deshawn's incorrect answer for the first problem, but now engages in the disagreement with Anaya's correct answer for the second problem. In her disagreement with 1/4 for the second problem, Kallie lays out the mathematical process that she goes through in making a proposal for 1 and 1/2, how she handles with mathematical issues she faces with, and how to support her claim (1 and 1/2) or refutes another possible claim (1/3). First, Kallie came up with the idea that adding a line changes the problem, so she tried to make sense of the problem as it is. Second, she takes into account one of the most common misconceptions about naming a fraction—only counting the number of pieces without considering what kinds of pieces—but eliminates this possibility by using Kadeem's idea. Third, she uses the idea proposed by Kadeem to justify her answer. In fact, Kadeem did not come up with the correct answer for his initial explanation, but his idea provides a critical tool for others to justify their ideas. Lastly, it is interesting to see that Kallie expresses her disagreement with Anaya's answer of 1/4, but she actually refutes 1/3 instead of 1/4 in her explanation.

Faced up with two conflicting ideas—the parts need to be equal but adding a line changes the problem, Kallie seeks the solution that resolves two conflicting ideas—identifying equal parts without adding a line—and negotiates herself to take a different whole. It is almost the end of session of Day 1, so the teacher wraps up the 20-minute of discussion about the brown rectangle problem by summarizing Kadeem's explanation, Anaya's explanation, and Kallie's explanation.

In the next day, on the first session of Day 2, the teacher invites Kallie on the board and re-launches a whole-group discussion about the second problem of the brown rectangle problem. After reminding that the students agree with the answer (1/3) for the first problem because of three equal parts and one of them is shaded gray, the teacher reviews Kadeem's idea (not equal) and Anaya's idea (adding a line), and then introduces Kallie's idea from the yesterday's class with annotating that Kallie has changed her mind.

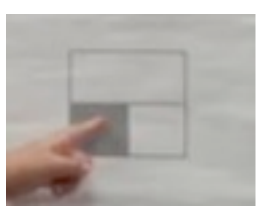
Before Kallie provides her explanation on the board, the teacher checks with Kadeem whether he changes his mind about the answer.

> So we have one-fourth might be an answer or no fraction, Kadeem, are you changed your mind about that? Or you're still saying that it's not a fraction?

As seen in the choice of modal verb of "might" in making a reference to 1/4 and not a fraction, the teacher opens the possibility to take Kallie's proposal of 1 and 1/2 seriously and indicates that the proposed answers are not rigorously reviewed yet. Because Kallie changes her mind while doing her homework the day before, the teacher checks with Kadeem again whether he changes his mind, but Kadeem stays with his original idea. Kallie explains her idea.

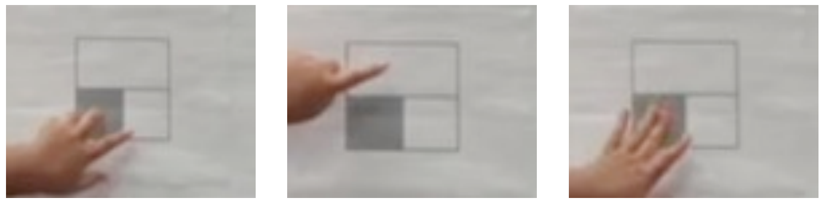
Kallie:	Well, I thought this fraction right here, at first it looked like one-third, but he-Kadeem, is it?
Teacher:	Kadeem.
Kallie:	Kadeem was right that it's not an equal fraction. So I was thinking since this (pointing to the shaded gray part, as shown in (a)) is divided up-this (pointing to the upper part of the whole rectangle, as shown in (b)) looks like one if you take this part (covering up the lower part of the whole rectangle, as shown in (c)) away, and this part (pointing to the shaded gray part, as shown in (d)) looks like one rectangle (pointing to the lower part of the whole rectangle, as shown in (e)) but it split in half. So I will call this part (pointing to the upper part of the whole rectangle, as shown in (f)) the whole and I will call this part (pointing to the shaded gray part, as shown in (g)), pieces, uhm, pieces of the whole, so the fraction I named is this one (pointing to the upper part of the whole rectangle, as shown in (h)) and one-half (pointing to the shaded gray part, as shown in (i)).

(a) (b) (c)



(d) (e) (f)

(g) (h) (i)



Similar to the initial explanation that she provided yesterday, Kallie refutes 1/3 using Kadeem's idea and then explain what she takes the whole and where the fractional part (1/2) comes from and where the whole number part (1) come from. Kallie then explains how she changed her mind while doing the homework yesterday.

Kallie: Yes, but I did change my mind while I was doing my homework last night and I realized that Phoebe, I'm sorry, uhm, she might be right, it is, it might be one-fourth because she can split it down in the middle.
Teacher: Okay, so you're-are you still, do you still want to say that this is one and one-half or do you want to say one-fourth?
Kallie: I'm going to say one-fourth.

Although Kallie changes her mind now, the teacher helps students understand why Kallie thought that it was 1 and 1/2. The teacher asks what Kallie called the whole for 1 and 1/2. Ahmed responds to the teacher's question.

Ahmed: She thought that the big rectangle was the whole because, if you take that line away from there, then if you take the line in the middle from those two squares away-
Teacher: -this right here?-
Ahmed: -that could be a rectangle.
Teacher: Uh-huh. Do you want to come up and show us what you mean? Or just want to talk from there?
Ahmed: I can talk from here.
Teacher: Okay. So say again if you take away this line
Ahmed: Yeah
Teacher: Then?

Ahmed: It could be a rectangle.
Teacher: Then say this (pointing to the upper rectangle) is the whole?
Ahmed: Yeah.
Teacher: And then what was she calling this (pointing to the shaded gray)?
Ahmed: A piece of the whole.
Teacher: And how much of the-how much was it? If this (pointing to the lower rectangle) is the whole, what was-how much is this (pointing to the shaded gray)?
Ahmed: Uh, that would be one-half.

After Ahmed's explanation, the teacher draws a line that Kallie calls the whole when she named for 1 and 1/2.

Figure 4.24. The whole that Kallie used for one and one-half

Because Kallie changed her answer now, the teacher asks what Anaya and Kallie now call the whole. Aryanna explains "Uhm, two-fourths in the whole that rectangle" Upon the teacher's request, Aryanna comes up to the board and traces the whole with her finger.

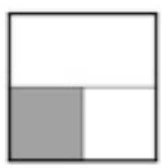
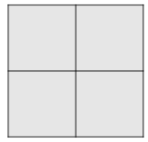


Figure 4.25. The whole for Anaya and Kallie agreed on for one-fourth

From these exchanges, the teacher lists the important things for naming a fraction on the board.

1. What is the whole
2. Divide into equal parts

In order to check whether the parts are divided equally, the teacher provides sticky gray squares so that the students could paste into the drawing, in which all the squares are the same size. David pastes all the sticky squares and explains "They are all same size and they are all equal."



Based on David's work, the teacher adds one more thing to the list: 3. Name one of equal parts. After 20-mintue of whole-group discussion, the teacher distributes another unequally partitioned rectangle problem and concludes the discussion about the brown rectangle problem.

Summary

The EML 2013 students initially propose two answers (1/3 and 1/2) for the first part of the brown rectangle problem and propose three answers (not a fraction, 1/4, and 1 and 1/2) for the second part of the brown rectangle problem. In explaining the brown rectangle problem on Day 1, the EML 2013 students initially miss the key idea of "equal" for the first part of the brown rectangle problem, use the pre-defined mathematical term without specifying the underlying concept behind the term (e.g., denominator, numerator); grant geometric names incorrectly, incoherently, and indistinguishably; and use the inaccurate language (e.g., "cubes") which its intended meaning is different from the accepted mathematical definition.

For the first part of the brown rectangle problem, Liberty proposes the correct answer (1/3) but provides the incomplete and partial explanation (i.e., missing "equal"). Liberty's partial explanation is elaborated by Deshawn's repeat, Liberty's repeat, Tina's add-on, Liberty's confirmation, Renee's agreement, and Ahmed's repeat. After hearing Liberty's explanation, Deshawn explains the incorrect answer (1/2). Deshawn's initial explanation is repeated by Mark, rephrased by the teacher, elaborated by Kallie, and revised by Deshawn. For the second part of the brown rectangle problem, Kadeem proposes "not a fraction" but addresses the key idea of "equal." The key idea of "equal" is emerged at the beginning of the whole-group discussion but it is readily pick up by the student, Anaya, to explain why it is important to draw a line and to propose her answer of 1/4. After hearing these two explanations, Kallie provides an explanation for 1 and 1/2.

To support students' development of mathematical explanation for the brown rectangle problem, the teacher distributes an equal opportunity to explain rather than heavily controlling the sequence of proposals; seeks for different answers; makes an extensive use of the initial explanation rather than eliciting various versions of explanations and just leaving them unexamined; requests for repeating and revoicing the initial explanation and then checks back with the initial explainer; invites students to the board so that the students supplement their verbal explanation with pictorial representations; delays the evaluation of the answer and the rigorous inspection of the completeness of explanation at the beginning; and increases the level of mathematical supports over time (no substantive mathematical supports, including clarifying what the rectangle refers to and remediating errors, during the set-up stage and the individual work).

4.7. Summary of the Chapter

In Chapter 4, I analyzed instructional interactions for teaching the brown rectangle problem managed by the same teacher, Ms. Ball, to five different cohorts of the EML students. As a series of developing the definition of fraction in different representation models (area model, set model, and number line model), the brown rectangle problem is first introduced to the EML students to develop the concept of making equal parts in a part-whole relationship of the area model. The brown rectangle problem consists of two sub-problems: naming an equally partitioned rectangle and naming an unequally partitioned rectangle. These two sub-problems play complementary roles in that the correct answer of the equally partitioned rectangle is used as a reference to make a proposal for the incorrect answer of the unequally partitioned rectangle whereas the correct answer but incomplete explanation of the equally partitioned problem is supplemented by the discussion about different proposals of the unequally partitioned rectangle. Because the incorrect answers play a key role for developing the core idea of making equal parts, the demand of preserving incorrect answers is high. The brown rectangle problem has demands in using accurate, correct, and coherent language, in building a connection between numerical representation and pictorial representation, and in building a logical structure of building an explanation. Thus, the demand of repeating and revoicing the initial explanation is high.

The difficulties that the EML students have in explaining the brown rectangle problem include:

- *Having difficulties with providing, hearing, and constructing an explanation in general*: Giving a public speech to a number of audience and being attentive to others' explanations are one of the most crucial elements to develop a mathematical explanation, but it is not an easy task for students to have confidence and fluency in them. Such problems include (1) declining the invitation to the board and refusing to explain in a public space; (2) having difficulties with speaking loudly enough and not listening carefully to what others say; and (3) appealing that they do not know how to explain or merely stating that they just know it.

- *Not establishing the mathematical grammar to describe the objects to be explained*: The language that an individual student uses to describe the brown rectangle problem varies and is inconsistent. For example, some students use the geometric names (e.g., three squares for the first problem; two squares and a rectangle for the second problem), but others use non-geometric names (e.g., three parts for the first problem; three parts for the second problem). In another example, the language to describe the right side of the rectangle varies such as a big rectangle, a big box, a half of box, the long part, full rectangle, the rectangle, half of the rectangle, and the third part. Using different languages, which are sometimes contradictory to each other, makes it difficult to communicate its intended meaning, thus being an obstacle to establish a common mathematical understanding.
- *Using inaccurate or incorrect language in which its intended meaning is different from the accepted mathematical definition*: Using accurate and correct mathematical language is an important part of developing a mathematical explanation, but the students have difficulties with using mathematically language in an accurate and correct way. This includes using the term "even" to indicate make the parts "equal," naming "a regular rectangle" to indicate "a non-rectangle square," "half" to indicate naming "one of two parts but not necessarily being equally partitioned," "the third box" to indicate the right side of the big rectangle, and "cubes" to indicate "squares."
- *Using pre-defined mathematical terms*: Having a fluency in using mathematical terms is an important aspect of developing a mathematical explanation. However, using pre-defined mathematical terms, such as denominator and numerator, but not making explicit the underlying concept behind such terms is one of the problems that individual students have.
- *Skipping the logical structure of naming a fraction or paying attention to the only partial components of naming a fraction*: The mathematical explanation for the second part of the brown rectangle problem has a logical structure but initial explanations that individual students provide often does not fulfill the logical structure of naming a fraction. For example, students explain "I counted" or "one

is shaded" or "I made two squares" without providing the full logical structure to arrive the answer.

- *Losing the purpose and focus of what is being explained:* When the explanation gets lengthier, students sometimes lose the purpose and focus of what is being explained.
- *Not building correspondences between an answer, an explanation, and representations:* The explanation provided is often disconnected with the answer proposed (providing a more general idea about a fraction rather than providing a specific explanation about the answer they propose); and is often disconnected with the pictorial representation (what the three refers to in the drawing and what the one refers to in the drawing).
- *Heavily using demonstrative pronouns*: In providing an explanation on the board, some students just point out the pieces in the drawing and heavily use demonstrative pronouns. From just hearing the verbal explanations that are heavily loaded with demonstrative pronouns, it is difficult for the audience to figure out which parts are referred to.
- *Grounding on non-mathematical reasons or procedural knowledge*: When the reasoning behind the answer was further asked, some students ground on non-mathematical reasons (e.g., I learned it in the fourth grade. My teacher told me.) or procedural knowledge (e.g., how to make equivalent fractions algebraically).
- *Missing the key definitional idea for mathematical explanation*: The key idea of "equal parts" are mostly missed in the initial explanations because students either do not recognize that "equal" matters for naming a fraction or they just take it for granted the idea.

CHAPTER 5.
CASE 2:
DEVELOPING MATHEMATICAL EXPLANATION FOR THE BLUE AND GREEN RECTANGLE PROBLEM

5.1. Overview

In this chapter, I analyze instructional interactions managed by the same teacher, Ms. Ball, for teaching the blue and green rectangle problem across five years (EML2007, EML2008, EML2009, EML2010, and EML2013). The blue and green rectangle problem is mainly composed of two parts: (1) naming a fraction for the blue triangle and (2) naming a fraction for the green rectangle. Building on the answers elicited from these two problems, the blue and green rectangle problem extends on the key idea for naming a fraction: whether or not two different geometric shapes—the blue triangle and the green rectangle—have the same fractional name. The concept of "equal" was developed by discussing the brown rectangle problem but it mainly considered equal shapes rather than equal size (area). As briefly described in Chapter 3, the blue and green rectangle problem has been used with slight variations in the layout of the rectangle (the rotation of the rectangle), the inclusion of written problem statement on the poster and in the handout, the wording of the problem statements, and the available resources, but the mathematical demand remains the same across five years. As an area model, the blue and green rectangle problem reinforces the concept of identifying the whole and elaborates the concept of "equal parts" into "equal area" for naming a fraction.

5.2. The Case of EML 2007

Preview

In the EML 2007, the blue and green rectangle problem is introduced on the first session of Day 7. After posting the drawing on the board, the teacher creates an opportunity for students to generate the problem statements. Several students attempt to make a problem statement but do not have much success in targeting the key ideas embedded in the blue and green rectangle problem. Faced with the students' struggles in generating problem statements, the teacher introduces two problems (naming a fraction for the blue triangle and naming a fraction for the green rectangle) on the board and let the students begin individual work. During individual work, the teacher nominates the third problem (naming a fraction for the blue triangle and the green rectangle together) on the board based on her observation of the student's work.

After 11-mintue individual work, the whole-group discussion begins by discussing the blue triangle. For the blue triangle, three answers (1/2, 1/5, and 1/8) are proposed by Mahluli, Ethan, and Daniel respectively. For the green rectangle, two answers (1/8 and 1/2) are proposed by Alexa and Hilaire, respectively. An extensive detailed analysis of the 33-minute instructional interactions managed by the teacher, Ms. Ball, for teaching the blue and green rectangle problem in the EML 2007 is provided below.

Extensive Detailed Analysis

After wrapping up a brief discussion about the warm-up problem of Day 7, the teacher makes a transition into the blue and green rectangle problem by putting up a poster on the board. In the poster, one triangle in the upper right corner of the big rectangle is shaded blue and one rectangle in the lower left corner of the big rectangle is shaded green, but the poster itself does not contain any written problem statements.

The teacher gives an opportunity for students to generate problem statements. Christopher initially proposed, “What is colored in it?” but revises it to, “What fraction is shaded in it?” Upon the teacher’s request to make the problem statement clear, Christopher further revises his previous problem statement to, “What fraction of the big rectangle is shaded?” Through the process of refining the problem statement, Christopher adds the language that signifies the reference of the whole (“big rectangle”) but he does not further clarify what he means by “the big rectangle.” Unlike the brown rectangle problem which is shaded in only one color, the blue and green rectangle problem is shaded in two colors. For that reason, the problem statement needs a clear identification about the color that is shaded in. After Christopher’s three trials of making a problem statement, the teacher gives a turn to Shane to further improve Christopher’s problem statement.

Teacher: What fraction of the big rectangle is shaded? Shane, can you help finish the question? What do I need to say, Shane? Is shaded what?
Shane: Is shaded in.
Teacher: But there’s two different shadings, so let’s make the question clear.
Shane: Is shaded in the square.
Teacher: No, that’s not I’m going to ask like.
Student: Equal square!
Teacher: Okay. (writing two questions next to the drawing)

What fraction of the big rectangle is shaded blue?

What fraction of the big rectangle is shaded green?

Student 1: Everything has to be the same length.
Student 2: I know the answer. That’s what I said.
Student 1: Yeah, everything has to be the same length.
Student 2: But all of that is not equal.
Student 3: No, you can trade it and make a different shape.
Student 1: How do you get traded it and make it a different shape?

Shane attempts to revise the problem statement offered by Christopher, but does not hit the targeted mathematical idea that the blue and green rectangle problem aims to reinforce. After spending about two minutes to generate the problem statements but noticing that the students do not have much success, the teacher introduces two problem statements to students. While the teacher writes the problem statements next to the drawing on the board, several students make comments, respond to each other, exchange an idea about what kind of mathematical work needs to be done, and express what kind of mathematical issues they are faced. Even though the language that these students used is neither accurate (e.g., "same length" instead of "same size" or "same area") nor detailed (e.g., the meaning of "equal"), these exchanges convey the key ideas for naming a fraction that the teacher aims to elicit in teaching the blue and green rectangle problem. Because they are not granted to make a proposal in a public space yet, the teacher does not pick up any ideas from these conversations.

In making problem statements, the students did not make much success at targeting the key mathematical point of the blue and green rectangle problem. The problem statements proposed by students are either non-mathematical questions, are lack of the necessary information, or expose too much information about what kind of mathematical work needs to be done. On the one hand, creating an opportunity for students to make problem statements in their own words could arouse their interest in the mathematical task and make them being well aware of the mathematical point embedded in the mathematical task. By being attentive to the problem statements that students generate, a teacher might be able to gain clues about the struggles that students have in accessing to the mathematical task. On the other hand, if students do not target the key point of the mathematical task for an extended amount of time, instructional time that needs to be invested for discussing the key ideas at a later point might be sacrificed at the entry point of working on the mathematical task.

After writing the problem statements on the board, the teacher asks students to read the problem statements and then asks students to work on the problem individually. While launching the blue and green rectangle problem, no further clarification about the problem statements is made in a public space. Given that "identifying the whole" was addressed by Lila's nomination of 1/2 for the second part of the brown rectangle problem on the previous day, leaving the interpretation of "the big rectangle" to the students provides an opportunity for a teacher to test how robust the students had established that concept.

During individual work, the teacher engages in the following mathematical work. First, the teacher provides mathematical supports for individual students upon a request, but does not resolve these issues in a public space. When Shane and Christopher request for the clarification about what "the big rectangle" means, the teacher traces "the big rectangle" with her finger for them individually, but does not further draw other students' attention to what "the big rectangle" means in a public space.

Second, a new problem statement is added based on the observation of the student's work. As written on the board, the blue and green rectangle problem is initially composed of two parts—naming a fraction for the blue triangle and naming a fraction for the green rectangle—as planned in the lesson plan. During individual work, the teacher notices that Doran produces only one answer, 1/4, in his notebook. One possible reason for producing 1/4 would be that Doran might take a part of the big rectangle as a whole (thus the big rectangle is composed of two wholes) instead of taking the big rectangle as a whole (thus the big rectangle is composed of only one whole). The teacher checks whether Doran's answer follows this line of reasoning, but Doran explains that he combines the blue triangle and the green rectangle together. The teacher points out that Doran answers a different question and asks him to write an answer for each of the two questions written on the board. After getting permission from Doran to share his idea with the class, the teacher pauses individual work and draws attention from other students.

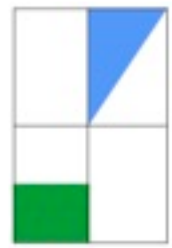
Teacher:	Okay. Doran just asked a third question that I'm going to add to the board. Doran, can you say what your question was altogether? Don't say what you think the answer is.
Doran:	What they are all together?
Teacher:	Good! If you put the blue and green altogether, how much is shaded out of the big rectangle?
Student 1:	I don't get it.
Student 2:	It's not equal.
Alexa:	Ms. Ball. It's not equal, how can we…
Teacher:	That's what you have to think about right now, Alexa. Okay, Alexa, because of the question I'm gonna say the question more clearly. Now I'm gonna say, put them together because you're worrying about that. I'm just gonna say, "how much of the big

rectangle is shaded all together?" (writing the third question on the board) Doran, that's the same question that you're asking? How much of the big rectangle is shaded all together?

1. What fraction of the big rectangle is shaded blue?

2. What fraction of the big rectangle is shaded green?

3. How much of the big rectangle is shaded all together?



Doran: Yeah.
Teacher: Okay.
Student 3: It's not equal. How do I...
Student 4: I don't know.
Student 5: It can't be equal.

While inviting Doran to share his idea, the teacher asks him not to blurt out his answer yet in a public space. In doing so, the teacher preserves a private space for other students to seriously consider the problem. As Doran's idea is added as the third problem on the board, several students make complaints about the new problem because the collective knowledge established from the brown rectangle problem—adding a line makes equal parts—does not resolve all of the issues they are facing in the blue and green rectangle problem. The knowledge of "equal" established from the brown rectangle problem is more like making "equal shape" but not fully extended to making "equal area" yet. The third problem would be accessible through the arithmetical approach of adding like fractions (1/8 + 1/8 = 1/4), geometrical transformation (cutting one figure to make another figure), or measurement concept (covering the whole with the unit). Without developing these understandings, students are likely to have difficulties in getting the correct answer for the third problem even if they succeed in naming a fraction for the blue triangle and naming a fraction for the green rectangle.

After the teacher adds Doran's question on the board, the class resumes individual work. During individual work, several students continue to seek for help solving this issue. Roddie addresses his difficulties in understanding the third question to the teacher.

Although the shape does not necessarily have to be the same to name a fraction, naming a fraction for two different shapes together is apparently challenging to Roddie, to whom it could be like adding an apple and an orange.

Roddie:	I don't get how I'm supposed to do number three. Because remember you said everything has to be the same size, you can't make the same size with two different shapes.
Teacher:	Hmm. So how much of the big rectangle is shaded blue? What's your answer to number one?
Roddie:	One-eight.
Teacher:	How did you get that?
Roddie:	Because … if the blue, if you just took out the green because you don't need the green, because it said blue. If you just make everything the same shape, you got eight shapes total and one is shaded.
Teacher:	And the blue, what did you think the blue was?
Roddie:	The blue is the same thing as the green.
Teacher:	So then what happens if you thought about how much was covered in the whole diagram? In the whole rectangle, you can't figure that one out?
Roddie:	(shaking his head)
Teacher:	So, that's what we'll find out in the discussion.

Expressing difficulties in understanding the third problem, Roddie makes a conjecture that, "You can't make the same size with two different shapes." His conjecture is the opposite statement that needs to be tested to get the correct answer. Neither evaluating nor testing the conjecture that Roddie made, the teacher checks what Roddie got for the blue triangle and what he got for the green rectangle. Roddie successfully names a fraction for each color shaded in (1/8 for the blue triangle and 1/8 for the green rectangle), but has difficulty with naming two shapes together because it requires more than just drawing lines. He chose the language "same size" but uses it as a synonym of "same shape." Instead of immediately resolving his issue with the third problem, the teacher checks Roddie's reasoning about the first two questions and postpones the discussion of the third problem for whole-group discussion.

The third question makes a bridge between the blue triangle and the green rectangle and creates a mathematically more challenging context, but obviously, it has a mathematically higher hierarchy than the first two questions. The blue and green rectangle problem does not just seek an answer for each color separately, but it has a purpose of connecting these two answers to elaborate on the concept of “equal parts” into “equal area.” Although the blue and green rectangle problem does not explicitly state, it implies the idea of proving how these two different shapes have the same fractional name.

Third, the teacher makes mathematical resources available for students to use. During individual work, the teacher writes the following working ideas of fraction on the board that were collectively developed by discussing the brown rectangle problem from the previous day.

Working ideas about fraction
1. identify the whole
2. equal parts
3. how many parts of the whole

Fourth, under the context in which collective resources are not available for use, the individual level of support for developing mathematical explanation is not the same as the whole-group level of support for developing mathematical explanation. The following excerpt illustrates the level of support given to an individual student in responding to the request for help in understanding the blue and green rectangle problem.

Shane:	I don’t get any of.
Teacher:	Okay. Were you able to figure out what this one is?
Shane:	Half.
Teacher:	How did you come up with a half?
Shane:	It’s half of this little rectangle (tracing the upper right side of the rectangle in his notebook).
Teacher:	Okay, and what part is it all of this (tracing the whole rectangle in Shane’s notebook)? Can you figure that out?
Shane:	Like a quarter?
Teacher:	You see if you can prove that. You know how to think about that. See if you can figure out.
Shane:	No, I don’t.
Teacher:	Well how would you draw the lines to help yourself think about that?
Shane:	I don’t know. This is just weird.

Shane expresses his difficulty with understanding the problem to the teacher, but immediately produces an answer with the teacher’s prompt (even though the initial answer is not a correct one). Upon the teacher's request for explaining, Shane identifies his reference of the whole as "little rectangle" which signifies a different size of the rectangle from the "big rectangle" in the problem statement. The language he used to identify his whole "little rectangle" does not match with the word "big rectangle" in the problem statement. Instead of pointing out the inconsistency between Shane's language and the language written in the problem statement, the teacher probes to find a relationship between "the little rectangle" that Shane pointed out and "the big rectangle" in the problem statement. Again, without much difficulty, Shane explains that "the little rectangle" is a quarter of "the big rectangle." Through the process of expanding the reference of the whole consecutively, Shane approaches the correct answer, a half of a quarter. Despite the teacher's further request for proof, Shane responds that he does not know how to prove. The teacher reminds him of the idea of drawing lines to make equal parts, but Shane still expresses his difficulty in understanding the problem. Shane does not record anything on his notebook. In this short excerpt between the teacher and Shane during individual work, the often-used discourse moves (e.g., repeating or revoicing) are not employed by the teacher. This provides insights about the potential role of collective resources in developing mathematical explanation and provides an opportunity to think about instructional resources other than discourse resources.

In the notebooks, seven out of 27 students made a clear record of 1/8 for each of the first two questions and two students produced an incomplete correct answer of 1/8. Eight students wrote 1/2 for each of the first two questions and two students wrote 1/4 for each of the first two questions. About two-thirds of students produced incorrect answers for the blue and green rectangle problem in their notebooks, but no remediation about errors were made during individual work. All of the students recorded the same answer for the blue triangle and for the green rectangle.

Table 5.1. Answers that the EML 2007 students wrote for the blue and green rectangle problem in their notebooks

Answers		Number of students
Same answer for each question	1/8	7 (Alexa, Kurtis, Lila, Marcel, Christopher, Pharell, and Roddie)
	Incomplete version of 1/8	2 (Doran wrote $\frac{1/2}{4}$ and Taleisha wrote $\frac{1}{4}$ in a half)
	1/2	8 (Amber, Asiya, Hilaire, Matthew, Niena, Ethan, Mahluli, Tatiana)
	1/4	2 (Daniel, Jaffa)
	4/8	1 (Micah)
	1 and 1/2	1 (Elis)
	3 and 1/2	2 (Iris, Avery)
Different answer for each question		0
Unrecognizable		0
No records		3 (Aiesha, Karen, Shane)

After an 11-minute individual work, the teacher reconvenes the class for a whole-group discussion. Instead of selecting a student who produces a particular answer, the teacher seeks for a volunteer who does not have a chance to explain in a public space yet. After waiting to see enough students' hands up, the teacher gives the floor to Mahluli to explain the first problem, naming a fraction for the blue triangle. At this moment, seven students (Marcel, Daniel, Asiya, Lila, Alexa, Kurtis, and Mahluli) raise their hands to explain. Kurtis, Lila, and Marcel recorded 1/8 in their notebooks, Asiya and Mahluli recorded 1/2 in her notebook, and Daniel recorded 1/4 in his notebook. A whole-group discussion begins with Mahluli's proposal of 1/2 for the blue triangle. Figure 5.1 shows what Mahluli wrote in his notebook.

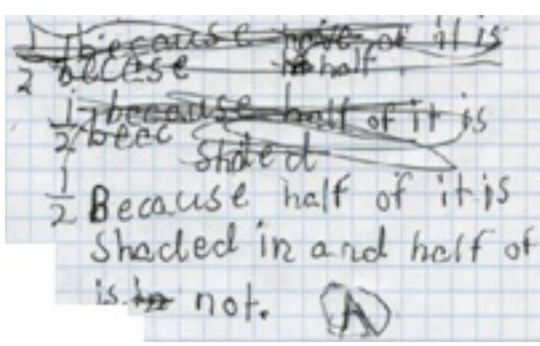


Figure 5.1. Mahluli's notebook writing for the blue and green rectangle problem

Mahluli:	Question one, I say it's one-half.
Teacher:	Okay. Can you explain how you came up with one-half?
Mahluli:	Because they both equal. They both equal, and one-half of it is shaded in and the other half is not. So that is…
Teacher:	Okay. Can you come up to the board and point and show us what you're looking at? Just-there's a diagram right there. Can you come up and show? Did everyone hear what Mahluli said? You should be thinking already about his reason. Who can repeat what Mahluli said? Okay. Well if you're listening carefully, you should always be able to tell what someone just said. Doran, what did he say?
Doran:	He said he's looking at the squ-rectangle, and he's saying it's one-half of the rectangle, not just- He's just- He's not looking at the whole, he's just looking at the one part-
Teacher:	Wait, wait, wait, let him talk. Don't go to explain it yet. Okay.

In responding to the teacher's request for explaining, Mahluli explicitly addresses one of the important ideas for naming a fraction (equal parts) within his reference. Mahluli does not make it explicit what he uses as a whole in naming a fraction, but considers the smaller rectangle in the upper right corner of the big rectangle as a whole rather than using the big rectangle as a whole. Because Mahluli provides an explanation from his seat, his explanation might not be clear for the audience. The teacher invites Mahluli to come up to the board and asks him to point out what he refers to. Meanwhile, the teacher asks other students to think about Mahluli's reasoning and to repeat what he said rather than analyzing, critiquing, agreeing, or disagreeing with Mahluli. The teacher asks Doran to repeat Mahluli's explanation. After identifying the whole that Mahluli considered, Doran provides a comment that Mahluli does not look at the whole, but just looks at one part of the whole instead. Even though Doran's comment is ultimately what the teacher aims to elicit, she does not rush in using his comment. Rather, the teacher does not let Doran go further than what Mahluli just explained and then gives the turn back to Mahluli.

Teacher:	Just a second. Everyone should be looking up at where Mahluli is pointing. Otherwise you won't understand his explanation. Shane, this way. Look up there. Okay?
Mahluli:	They both equal, and half of it is shaded in. So that makes it one, one-half.
Teacher:	Okay. So let's look at our working ideas about fractions that we were doing earlier today. Can someone say what, or maybe you

should say what are you calling the whole? When you're looking at the whole what you are looking at?

Mahluli: The whole. The whole square.

Teacher: Can you put your finger around the part you're calling the whole?

Mahluli: (tracing the rectangle on the upper right corner of the big rectangle) The whole.

Teacher: Okay. So, do you see where he just pointed?

Students: Yes.

Teacher: Okay. And where are the equal parts? Can you show us the equal parts?

Mahluli: These two (pointing the blue triangle and the white rectangle that are both on the upper right corner of the big rectangle).

Teacher: Okay. And how many parts are shaded?

Mahluli: One.

Teacher: Okay. Raise your hand if you understand what Mahluli did. Who knows what Mahluli did to get his answer of one-half? I don't want to hear how you agree or disagree. I just want you to tell me what did he do. Kurtis?

In giving the turn back to Mahluli, the teacher draws other students' careful attention to what Mahluli will point at. Due to the complexities in describing the objects to be explained, it is crucial to build a connection between the verbal explanation and the pictorial representation. Mahluli repeats his initial explanation, without much being influenced by Doran's additional comment. After Mahluli re-explains his answer, the teacher turns students' attention to the working ideas of fraction that the EML 2007 cohort collectively developed and asks Mahluli to identify the whole that he considered in producing 1/2. In responding to the teacher's request to identify the whole, Mahluli uses the language of "the whole square" and then traces the rectangle on the upper right corner of the big rectangle. After checking that everyone is able to see Mahluli's reference about the whole, the teacher addresses another working idea of fraction—making equal parts. The teacher checks again whether other students hear Mahluli's explanation, without agreeing or disagreeing, and asks Kurtis to repeat Mahluli's explanation.

Kurtis: He just made the part where the blue part is shaded. He just used that rectangle as a whole.

Teacher: Okay. Let's draw it on here so we can keep our original picture. You used this to be the whole, right (drawing a line with red marker what Mahluli mentioned as a whole)?

Mahluli:	Yes.
Teacher:	Can everyone see this?
Students:	Yes.
Teacher:	And what did he do then, Kurtis?
Kurtis:	And then he had saw that one part was shaded and the other part wasn't so he…
Teacher:	And are these two equal parts? So if Mahluli calls this the whole (writing 1/2 on the board), is he right that that's one-half?
Students:	Yes.

After hearing Mahluli's explanation, the teacher asks other students to repeat what Mahluli did, without indicating whether they agreeing or disagree with Mahluli's idea. Kurtis uses vague expression but describes Mahluli's work as considering "the part." The teacher draws the boarder with a marker and checks it with Mahluli. And then Kurtis continues to explain the shaded part and the unshaded parts. Adding "equal" on Kurtis' statement, the teacher asks other students whether Mahluli's answer would make sense if he uses a different whole.

At this point, it is interesting to compare the work between responding to Mahluli's idea during a whole-group work and responding to Shane's idea during individual work. Both students come up with the incorrect answer of 1/2, but the instructional supports provided by the teacher are different. Table 5.2 compares instructional supports provided to Shane during individual work with instructional supports provided to Mahluli during whole-group work.

Table 5.2. Comparison between instructional supports provided to Shane during individual work and instructional supports provided to Mahluli during whole-group work

	Responding to Shane's idea	Responding to Mahluli's idea
Setting	• Individual work	• Whole-group work
Answer	• 1/2	• 1/2
Reasons explained by the student	• "It's half of this little rectangle"	• "They both equal a one, one-half of it is shaded in and the other half is not."
Instructional supports provided by the teacher	• Locating Shane's reference of the whole onto the intended reference of the whole	• Inviting Mahluli to the board • Asking audience for attentive listening • Asking other students to repeat but not to evaluate • Asking for using the visual supplements (e.g., pointing to the pieces by tracing the boarder) • Using the working definition of fraction • Not locating Mahluli's reference of the whole onto the intended reference whole
The trajectory of explanation	• Shane's initial explanation → Shane's further explanation → Shane's struggles with making the connection between these two explanations	• Mahluli's initial explanation → Doran's repeating (but more like analyzing than describing) → Mahluli's repeating → Mahluli's supplementary explanation by using the working ideas of a fraction → Kurtis' interpretation → Checking with Mahluli → Kurtis' explanation

Even with the same incorrect answer, the explanation provided by Mahluli is different from the explanation provided by Shane. Shane addresses the reference of his whole, but Mahluli addresses that the parts are equal. This might lead to providing different instructional supports by the teacher, but the context in which collective resources are available for use during whole-group would set up different instructional supports for Mahluli. In contrast to the instructional supports provided to Shane during individual work, the teacher makes use of repeating and supports to build a connection between the verbal representation and the pictorial representation but does not extend the partial reference of the whole to the intended reference of the whole.

After utilizing the working definition of fraction to make sense of Mahluli's answer, the teacher points out that Mahluli takes something different as the whole and then clarifies what "the big rectangle" refers to in the problem statement. Avery, who wrote $3\frac{1}{2}$ in her notebook, reads aloud the problem statement, comes up to the board and then traces the big rectangle with her finger. As Avery comes to the board, the teacher draws Mahluli's attention to Avery's presentation.

> Teacher: Okay. The whole big rectangle. Okay. So now I need someone to explain, if you look at the whole big rectangle as the whole, okay, now we want to talk about all of this. (drawing a line with red maker of the big rectangle) The question asks, if you use the whole big rectangle to be the whole, how much is shaded blue? Mahluli, do you see the difference between the question you answered and this question? Okay. What's the difference?
>
> 
>
> Mahluli: You gotta try to figure out, out of the whole square.
> Teacher: Out of the whole rectangle. And you used what?
> Mahluli: And I did half of the rectangle.
> Teacher: You did a smaller part of the rectangle. Okay?

The language that Avery used (the whole rectangle) is similar to the one that Mahluli used (the whole square), but it appears that they consider a different part of the big rectangle. After identifying the big rectangle as being intended in the problem statement, the teacher checks with Mahluli whether he sees the difference between his reference and Avery's reference. Mahluli explains that he is supposed to use "the whole square" but he used "half of the rectangle" to produce an answer for the blue triangle. His language choice is inaccurate because he uses "the quarter of the rectangle" not "the half of the rectangle." The mathematical object, the smaller rectangle on the upper right corner of the big rectangle, is named differently by the students but is progressively refined through the collective effort (e.g., “it” by Mahluli→ “the rectangle” by Doran→ “the whole square” by Mahluli→ “the part” by Kurtis→ “half of the rectangle” by Mahluli→ “a smaller part of the rectangle” by the teacher).

The mathematical work engaged by the teacher includes (1) the continuous involvement of Mahluli in developing a mathematical explanation; and (2) correcting inaccurate language use so that students are able to appropriate the accurate term. First, the teacher actively engages Mahluli, who proposed the incorrect answer, in the construction of a mathematical explanation rather than making him feeling alienated or excluded. When Kurtis provides an interpretation of the whole that Mahluli used, the teacher checks with Mahluli whether Kurtis captures his thinking well. When Avery identifies the intended whole in the problem statement, the teacher draws Mahluli’s attention to Avery and gives Mahluli another turn to explain the difference between Avery’s reference of the whole and Mahluli’s reference of the whole.

Second, the teacher corrects the inaccurate language that students use so that they are able to appropriate the accurate terms, without having any substantial deviations from the main ideas for naming a fraction. For example, the teacher corrects the language from “square” to “rectangle,” from “the whole square” to “the whole rectangle,” and from “half of the rectangle” to “a smaller part of the rectangle.” Influenced by the teacher’s language use, the students are able to appropriate the language as used by the teacher.

After clarifying the reference of the whole that Mahluli used, the teacher moves onto eliciting another answer for the blue triangle by providing a clear reference of the intended whole.

Teacher:	So, who thinks they can explain what part of big rectangle is shaded blue? I should see more hands than two people’s hands. What fraction of the whole rectangle is shaded blue? Ethan?
Ethan:	Fifth.
Teacher:	Fifth? Can you come up with explain how you got fifth?
Ethan:	Mm, hmm.
Teacher:	Is that what you said fifth?
Ethan:	Yeah.
Teacher:	Okay, how did you get it?
Ethan:	Uh-uh... (silence for few seconds) No.

Teacher: First of all, you identify the whole. Right? The whole is all of this. (tracing the whole big rectangle with her finger on the board). Okay, now, next what did you do? Where are the equal parts?

Ethan: (silence for a few seconds)

Teacher: Or how did you get the fifth?

Ethan: Well, I know half of fourth was fifth.

Teacher: Okay, and then what?

Ethan: (silence for a few seconds)

Teacher: Where are the half of fourth coming?

Ethan: Uh.

Teacher: Where did you see fourth in the drawing?

Ethan: Four squares.

Teacher: Okay, four squares makes fourth. It’s really rectangles. So, are you seeing this one, this one, this one, and this one? (tracing each of four smaller rectangles on the board)

Ethan: Yeah.

Teacher: So, did you see what Ethan see four equal rectangles, Mahluli and Doran? One, two, three, four. (pointing each of four smaller rectangles on the board) And then what? And then you looked to the blue and you said it was half of one of those, right?

Ethan: Yeah.

Teacher: And then you called it?

Ethan: Fifth.

As shown in Figure 5.2, Ethan initially recorded that “1/2 is shaded blue of one square” and “1/2 is green in the other square” in his notebook during the individual work. He does not specify which “square” he refers to, but the words that he chose (“one square” and “the other square”) imply that he considers the smaller rectangle circumscribing each shaded part as a whole. Ethan recorded the same answer as Mahluli, but does not back up Mahluli’s proposal. Instead, he makes another proposal of 1/5.

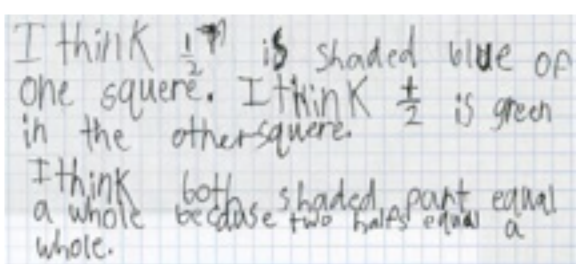


Figure 5.2. Ethan’s notebook writing for the blue and green rectangle problem

After hearing Mahluli's explanation and Avery's clarification about the big rectangle, Ethan might acknowledge that he took a different whole than as intended in the problem statement during individual work and then change his mind from "1/2" to "half of fourth." Ethan understands the relationship between a part and a whole and identifies equal parts correctly, but is not equipped with mathematical knowledge to handle "a fraction of a fraction" arithmetically. He proposes 1/5 as the answer but declines the teacher's invitation to the board and shows difficulty in explaining about his proposal. Facing Ethan's declination, hesitation, and silence, the teacher not only uses sufficient waiting time to elicit an explanation, but also uses the working definition of fraction as a resource to support Ethan in explaining his idea. After checking the whole that Ethan used, the teacher asks Ethan about the second working idea of fraction (identifying equal parts). As Ethan maintains his silence, the teacher makes a detour to understand where fifth comes from. Ethan answers that he knows that half of fourth is fifth—which is a mathematically incorrect claim. The teacher then proceeds to the next step but Ethan keeps his silence again. Instead of checking whether he makes equal parts to identify the fraction or unpacking the fifth further, the teacher attempts to clarify where half of fourth comes from. Because of Ethan's difficulty in identifying the half of fourth, the teacher narrows the scope of investigating Ethan's reasoning for where the fourth comes from. It is interesting to see that the teacher starts to unpack where the fourth comes from rather than starting to unpack where the half comes from. One possible reason might be that the class already discusses how someone might call one-half from Mahluli's proposal, but does not have access to hear how someone might see one-fourth. Another reason might be that unpacking one-fourth first makes it easy to keep the intended whole in the problem statement.

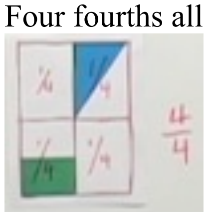
Ethan finally provides the source of his reasoning about the fourth. After quickly correcting the language from "four squares" to "four equal rectangles," the teacher makes clear that Ethan uses four equal rectangles, while keeping track of Mahluli who made an incorrect reference about the whole and Doran who comments on Mahluli's explanation. The teacher counts four equal rectangles and confirms with Ethan that he sees the blue triangle as "half of one of those four rectangles" but calls fifth. Ethan's answer is mathematically interesting. His approach to naming a fraction for the blue triangle is mathematically reasonable but he, as well as the class, does not establish the concept or algorithm to figure out a fraction of a fraction. One-half of one-fourth could be solved through multiplication of fractions ($\frac{1}{2} \times \frac{1}{4}$) or through a compound fraction ($\frac{1/2}{4}$). He takes a mathematically valuable approach but does not equip the mathematical knowledge to proceed with multiplication of fractions arithmetically.

After re-checking that Ethan names the blue triangle as half of four equal parts and calls fifth, the teacher straightforwardly addresses the mistake Ethan made and asks other students to help fix his mistake, but neither pursues further how he could get one-fifth from half of fourth nor makes advance into how to translate "a half of fourth" in an algebraic expression at this moment. In case of Mahluli, the teacher did not state that Mahluli made a mistake in his proposal of 1/2 and encouraged other students to understand Mahluli's reasoning. In contrast, with Ethan, the teacher states that Ethan made a mistake in his proposal of 1/5, reviews what Ethan did, and then asks other students to fix it.

Teacher:	Ethan did something very systematic about the way he thinks, the way he thought, but at the end, he made a mistake. But, it's only at the very end. So wonder whether somebody can help use what Ethan did and then fix the last step. So, let's start over. Ethan? What did you call the whole?
Ethan:	The…
Teacher:	What were we looking to call the whole? Big rectangle, right?
Ethan:	Yeah.
Teacher:	Then what did you see the next?
Ethan:	Four smaller rectangles?
Teacher:	Four small equal rectangles. One. I am not gonna, I marked them I guess. Like that? It that right?
Ethan:	Yeah.
Teacher:	And you call each one of them what?
Ethan:	Fourth.
Teacher:	Each one of them, he called one-fourth (writing 1/4 in each rectangle). Why would we call each one of them one-fourth? Can someone explain? Why would each one of those parts be called one-fourth? Amber, can you explain why?

Instead of immediately fixing the mistake, the teacher gives Ethan an opportunity to go through the process again. Using the working definition of fraction that is available on the board, the teacher supports Ethan in identifying the whole and showing equal parts. Even though Ethan's proposal of 1/5 is neither mathematically correct nor accurate, his proposal contributes to making a correct reference to the intended whole while keeping the idea of making equal parts. While Ethan explains his ideas, the teacher fills the missing word in his explanation. More specifically, Ethan said that he identifies "four smaller rectangles" but the teacher expands on this explanation by adding "equal" to his initial explanation. Similarly, when Ethan calls one of four equal rectangles as fourth, the teacher corrects it as one-fourth. Such minor revision of the words does not change the flow of instructional interactions but creates a context in which students could appropriate the accurate mathematical terms. The teacher then asks Amber to explain why they call one of four equal parts as one-fourth.

Amber:	(after keeping silence for a few seconds) because… hmm… there are, each one equal part of the four
Teacher:	Each one out of the four equal parts. One-fourth, one-fourth, one-fourth, and one-fourth. How many fourths are all together in the whole?
Students:	Four fourths.
Teacher:	Four fourths all together. (writing 4/4 on the board)



Amber, like Mahluli and Ethan, initially recorded 1/2 in her notebook during individual work, but is able to explain why the smaller rectangle that circumscribes the blue triangle is 1/4.

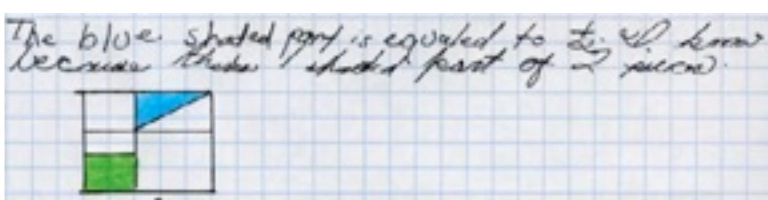


Figure 5.3. Amber's notebook writing for the blue and green rectangle problem

Amber makes explicit the concept of equal parts to name a fraction. The teacher makes a minor change to Amber's explanation by changing the order of the words ("each one equal part of the four" to "each one out of the four equal parts") and counts each rectangle by 1/4 to illustrate how much the whole is. By doing so, the teacher brings forth an idea that the whole corresponds to 4/4. After Amber's explanation about how Ethan sees 1/4 from the drawing, the teacher gives Ethan another turn.

Teacher:	Ethan, keep going. Then you said blue is what?
Ethan:	Fifth.
Teacher:	No, before you named it. What did you say it was?
Ethan:	(silence)
Teacher:	You said, in a relation with fourth, what did you say it was?
Ethan:	A half?
Teacher:	A half of fourth. Is he right half of fourth?
Students:	Yes.
Teacher:	So the place he went wrong was at the very last step. What did he do? What did he do for the last step? What did he call it? Doran?
Doran:	Five fourth?
Teacher:	No, he didn't call it five fourths. What did he call it? But, what name did you give, Ethan?
Ethan:	Fifth?
Teacher:	He did give a name of fifth. But, it's not a fifth. So, we need to figure out what we call this, if this is half of fourth. So anybody has a way that we can figure out what to call that? How could we figure out what to call that? Each one of big rectangles is fourth. Okay? That's what Ethan reasoned. But he came to call it to something. He called it one-fifth. We need to figure out what to call, not to call one fifth. Daniel?

The teacher gives Ethan a turn again to review what he calls the blue triangle in relation to 1/4 but does not ask how Ethan gets 1/2 from the drawing. One reason might be that the idea of 1/2 is already extensively examined by Mahluli's proposal. The teacher confirms with students that it makes sense to call the blue triangle as "half of fourth" and directly addresses that the answer is not 1/5. The class approves the idea of calling the blue triangle as 1/2 of 1/4 but looks for a name to represent it as a single fractional amount. Daniel, who initially wrote 1/4 in his notebook, gets a turn to explain. Again, the teacher does not pursue how to translate "half of a quarter" in a single fractional amount algebraically.

Daniel:	One-eighth.
Teacher:	How did you come up with one-eighth?
Daniel:	Oh, it's actually not maybe one-eighth. Two-eighths.

Teacher:	What's two-eighths?
Daniel:	Uhm, the half of the-
Teacher:	We only talk about the blue right now.
Daniel:	The blue right now, the blue would be one-eighth.
Teacher:	So, can you explain how you got one-eighth to be the name?
Daniel:	Two-eighths.
Teacher:	How were you getting eighth? Let's just figure out how you get the eighth at all.
Daniel:	Well, because if you cut it, if you're cutting one fourth and half, you will be getting, you have to get, you have to get, the bottom have to be mores, it will be given more, and the top of it will be given one more.
Teacher:	(drawing diagonals in each rectangle) Someone-you want to come up and count and show us you're getting the eighth there?

Daniel records 1/4 in the notebook during individual work, but correctly names the blue triangle as 1/8 during a whole-group discussion. The teacher asks Daniel how he comes up with 1/8, but Daniel changes his mind to 2/8. Before Daniel fully explains how he gets 2/8, the teacher clarifies that they are looking at the blue triangle only. It is unclear whether Daniel names the blue triangle and the green rectangle together as 2/8 or whether he names the blue triangle and the white triangle together as 2/8, but it is clear that he looks at more than the blue triangle. After being clearly directed to look only at the blue triangle, Daniel changes his answer to 1/8 but goes back to 2/8 again. To prevent the further complexity or distraction, the teacher sticks to considering how to get 1/8 first from the diagram. In Daniel's explanation, he mentioned about cutting each one-fourth into half but mainly focuses on the number of the numerator and the number of the denominator by describing it as getting more and more. Daniel conveys the idea of cutting each one-fourth into half but does not mention making eight equal parts. The teacher draws diagonals in each rectangle and asks Daniel to count each part by one-eighth.

Daniel:	(come to the board) Well, if you look at-well, one fourth right here would be, if you add up to, you have to double it, cause two pieces

right now, so you double this (pointing to the numerator) and double this (pointing to the denominator), so when you add this, double it, so you have two here, and then you have eight right here, so when you show, one eighth for that. Blue is one eighth and white is one eighth too. So, the other would be-it would be equal-whole square is equal to two eights.

Teacher: Daniel, it's hardly to hear you. Can you speak up?

Daniel: Well, this, right here would be one-eighth (pointing to the blue triangle) and this right here would be one-eighth (pointing to the white triangle next to the blue triangle), so the whole shape is, I think-no, I'm now confused.

Daniel accepts the teacher's invitation to the board to explain how he gets 1/8 from the drawing. He starts with each small rectangle (one-fourth) and then expands his explanation that the upper right corner of the big rectangle equals two eighths because the blue triangle is one-eighth and the white triangle is one-eighth. Daniel's identification of the upper right corner of the big rectangle as 2/8 might be used to reinforce the concept of equivalent fraction because the class already establishes knowledge that each small rectangle is 1/4. Instead of making the issue more complicate or extending to the equivalent fraction, the teacher keeps the focus on the eighth and eight equal parts. Instead of further probing Daniel's idea, the teacher gives credits him for dividing the whole into equal shapes and then lets other students to take over Daniel's idea.

Teacher: So, Let's someone to take over for you've gotten so far. You gave us the idea of cut this up. Let's do someone else to take over here. Marcel?

Marcel: It's cut like this, so it's one shaded, one out of-and if you count all of this groups, one, two, three, four, five, six, seven, eight, and eight groups and one of them is shaded.

Teacher: Okay. Did you see what Marcel did? He counted eight equal parts, and how many of them are shaded blue?

Student: One.

Teacher: One. Let's count them again. One, two, three-I don't want to count them. I want you count them. One- (pointing each triangle)

Students: One, two, three, four, five, six, seven, eight.

Teacher: Okay, those are equal, four equal-eight equal parts and how many are blue?

Student: One.

Teacher: So, what do we write for that?

Student: One-eighth.

Using Daniel's idea of "cutting it up," Marcel identifies eight parts. In repeating Marcel's explanation, the teacher adds the language of "equal" and counts eight equal parts one by one with the students and confirms that only one of them is shaded in blue, leading to the answer of 1/8. The teacher returns to Ethan, who initially called half of a fourth as fifth, to correct his mistake.

Teacher:	One-eighth. So, Ethan, can you explain now where your thinking went wrong? Cause most of your thinking was right. So, where do you think your thinking go wrong? Can you say?
Ethan:	Uh…
Teacher:	What did you call that piece?
Ethan:	A fifth.
Teacher:	And why would it be called one-eighth, not one-fifth?
Ethan:	Because all… of those, fourth, diagonal
Teacher:	And how many parts do you get then?
Ethan:	eight.

Ethan's explanation was quite incomplete in making a connection between fourth, diagonal, and one-eighth at the beginning but he can now identify eight parts. Instead of neglecting or rejecting Ethan's incorrect answer, the teacher values Ethan's proposal and encourages other students to understand his reasoning behind 1/5. In addition, the teacher keeps Nathan involved in developing a mathematical explanation. After checking Ethan's understanding, the teacher moves on to exploring Ethan's reasoning behind 1/5.

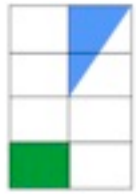
Teacher:	Okay, so Ethan did something very helpful. This morning, I would like to compliment for that. He thought about the problem very carefully, and you made a mistake, that's very natural. Can you think of any reason why somebody would call it one-fifth? We know when we name fraction, we were supposed to be how many equal parts and then figure out how many parts of the whole. But why did Ethan call one-fifth? Can anybody think about reason? It's a reasonable mistake to be made. It's not correct, but it's reasonable what he did that. What was his thinking might have led him to one-fifth. What do you think, Christopher?
Christopher:	Like, because… the ones that are colored, like, I don't think he counted one's that colored, I think that he counted the blank ones cause they would be five of them. Because it's the only one I don't have any color in them. That's completely blank in the drawing.
Teacher:	Maybe he counted the blank ones. I think there might be another reason. Do you know how you got the one-fifth? Anyone had

	thought about why it might be sensible to think one fifth? Matthew, what do you think?
Matthew:	He probably, uhm, count he just the, uhm, other rectangles, how they, uhm, where he counted those two, uhm, one rectangle and then counted one shaded.
Teacher:	Ah, ha. How did you get five from them?
Matthew:	Because if you count [inaudible] that is three, if you count those two, that's five.
Teacher:	Okay, that's possibility. He might also thought about it, because it was cutting one fourth and half, it might be thought five is the next. It might be another way. That's you thought, wasn't it? That's reasonable, but that's not how name fractions. We name fractions by counting how many equal parts we have, after we did that, you have equal parts, so it's one-eighth.

It is important to note that both Christopher's interpretation and Matthew's interpretation ignore the concept of equal partitioning, but Ethan's reasoning preserves the concept of equal partitioning. The teacher admits that these are other possibilities, but adds her interpretation of Ethan's faulty reasoning as caused by the incorrect way of counting fraction, which treats half of a fraction as the next counting number for the denominator instead of doubling the number in the denominator.

After clarifying how to get 1/8 for the blue triangle problem, the teacher moves to eliciting an answer for the green rectangle problem. Alexa proposes her answer of 1/8 by making a reference to the answer for the blue triangle.

Alexa:	I think it's the same as number one because it's half of, it's still half, but it's not a triangle, it's square and—
Teacher:	You can come up and split it up, so you can show it? I'll give you a blank one. Alexa says it's the same reasoning and it's the same answer. But, the picture gonna look different, right? First of all, what's the whole?
Alexa:	The whole is the whole rectangle. (tracing the big rectangle with the marker)
Teacher:	Okay, everyone is with her, so far?
Students:	Yes.
Teacher:	Watch and she gonna show you that the answer for question two is also one-eighth.
Alexa:	I think one-eighth because I just split, like this (drawing lines), and it's four of them, and it's (counting the little rectangles after splitting)

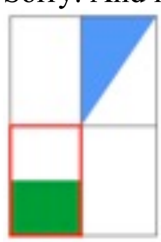


Teacher:	How many equal parts do you have now, Alexa? Can you count them for us?
Alexa:	One, two, three, four, five, six, seven, eight. (counting each small rectangle) It's eight and one shaded in. So, I thought it was one eighth because it's the same thing as this (pointing to the blue triangle). It's not a triangle. It's square. So, I was counting the whole as the rectangle. (tracing the big rectangle with her fingers) And.. uh… the whole.. would.. there're all equal parts because they are all squares.
Teacher:	They are actually rectangles. Right? They are not, they don't have equal size. So rectangles. Okay? I think that this is a little bit longer than this way. So, they are not squares. So, they call it rectangles. But keep going. How many equal rectangles do you have?
Alexa:	I have eight equal rectangles in and one is shaded in, so one-eighth.
Teacher:	Can you write the number one-eighth next to your drawing?
Alexa:	(writing 1/8)
Teacher:	Okay, Alexa reasoned about that well.
Student:	I agree.
Teacher:	She counted it up to eight parts and showed that one of them is shaded.

Alexa makes a reference to the answer for the blue triangle and identifies that the green rectangle is half of the rectangle. Because of the complexities involved in providing a clear explanation about the accurate reference, the teacher invites Alexa to come up to the board to explain her idea and asks Alexa to point out what she refers to. Before Alexa provides her explanation again in front of the class, the teacher asks her to identify the whole. After tracing the big rectangle as a whole with her finger, Alexa cuts each of four rectangles into half. The teacher follows up with Alexa on how many equal parts are produced. Alexa counts eight small rectangles after making the split, makes a reference to the answer for the blue triangle, identifies the whole by tracing the big rectangle with her finger, and adds that all parts are equal. The teacher points out that the eight parts are rectangles, not squares, because the length of sides are not all equal and then asks Alexa to continue with her explanation. Even though Alexa names the parts as squares but not as rectangles and repeats some of her explanation, it is quite a complete explanation by utilizing all of components in the working definition of fraction.

After reviewing Alexa's reasoning, Hilaire raises her hand to propose another answer for the green rectangle. Although the teacher did not call for another possible solution for the green rectangle, Hilaire voluntarily shares her idea with the class.

Hilaire: I just thought that it was one-half because it's only going that—which was shaded green and how much was it. I just thought that you are using the one block in the half of that is.
Teacher: Okay. Did everyone understand what Hilaire said?
Students: Yes.
Students: No.
Teacher: Okay. Say one more time. Very good, Hilaire.
Hilaire: I thought that it was one-half because I thought it was just saying what is the amount shaded with the green and I thought that she was just using the block that has green in it and split that into half, so that would be one-half.
Teacher: So Hilaire says she thought it was asking about green, so I think that this is what you are saying. You thought it was asking about this. (drawing a line of the lower left side of the rectangle). Oops. Sorry. And it's half of that?



In contrast to the tense that other students used in proposing the answers, Hilaire used past tense to share her idea of 1/2. In addition, Hilaire's reasoning is quite parallel to Mahluli's reasoning of 1/2 for naming a fraction for the blue triangle but Hilaire more clearly states the reference of the whole she used in naming the green rectangle as 1/2. After Hilaire's explanation, the teacher checks whether other students understand her explanation and then asks Hilaire to repeat her explanation. The teacher repeats Hilaire's explanation and marks the whole that Hilaire used and asks other students what Hilaire did. While making a reference to the working definition of fraction, the teacher asks Taleisha what Hilaire considers as the whole and asks Niena how many equal parts are and how many parts of the whole is shaded. Afterwards, the teacher asks students what is supposed to be the whole in the problem statement. Lila states that the big rectangle should be the whole and comes to the board to trace it. The whole-group discussion about the blue and green rectangle problem is wrapped up by reviewing two important ideas for naming a fraction—identifying the whole and making equal parts—and then moves to another mathematical task.

At the beginning of the second session of Day 7, the teacher revisits the blue and green rectangle problem. The session begins by reviewing the three important ideas for naming a fraction that are written on the board: (1) identify the whole; (2) equal parts; and (3) how many equal parts out of the whole. After briefly reviewing the working ideas for naming a fraction, the teacher reviews two different explanations for each problem. The teacher summarizes the first explanation which takes the big rectangle as a whole and then divides into eight equal parts and names the shaded part as 1/8 because one of eight equal parts is shaded. The teacher then brings up another idea that someone took a different whole and named it as 1/2—Mahluli named the blue triangle as 1/2 and Hilaire named the green rectangle as 1/2—and asks students to explain why it is okay to name 1/2 to clarify the reference of the whole.

Summary

The EML 2007 students propose three answers (1/2, 1/5, and 1/8) for the blue triangle and propose two answers (1/8 and 1/2) for the green rectangle. While launching the blue and green rectangle problem on Day 7, some of the EML 2007 students blurt out the key idea of naming a fraction (i.e., equal) and address their concerns that the blue triangle and the green rectangle are not equal (shape). In explaining the blue and green rectangle problem on Day 7, the EML 2007 students initially have initially developed a strong sense of making equal parts to name a fraction but does not explicitly address the whole; use the language incorrectly, inaccurately, and inconsistently; use the pre-defined mathematical term without specifying the underlying concept behind the term (e.g., denominator, numerator); grant incorrect geometric names; do not build a connection between the numerical representation and the pictorial representation; lose the track of what is being explained (e.g., Daniel's confusion between 1/8 and 2/8); and have lack of sufficient mathematical knowledge to translate "a fraction of a fraction" into a single fractional amount. However, the EML 2007 students evolve mathematical ideas instead of holding up their initial answer. For example, Ethan, Amber, and Daniel recorded the incorrect answers by taking the intended whole incorrectly. Instead of backing up Mahluli's proposal of 1/2, they make different proposals.

To support students' development of mathematical explanation for the blue and green rectangle problem, the teacher does not take up unauthorized ideas blurt-out at the beginning of the lesson so that all students have sufficient intellectual space to wrestle with the idea; does not rush on using the key idea offered by an individual student at the beginning of the lesson under the context in which the whole class does not establish the accessibility to the key mathematical idea yet (e.g., Doran's interpretation about Mahluli's work); provides mathematical supports upon an individual student's request but does not address the issues in a public space; does not jump into the mathematical knowledge that the class has not yet established (e.g., 1/2 of 1/4); invites the students to the board to allows them to clarify what is being explained and to build a connection between a verbal explanation and a pictorial representation; uses sufficient waiting time and reformulates the questions according to the students' needs when faced with the students' declination, hesitation, shyness, silence, and struggle to provide an explanation; constantly keeps students who propose incorrect answers engaged in the discussion instead of making them alienated or excluded; uses the working definition of fraction to support students to explain; asks the students neither to agree nor to disagree with the key correct answer; and replaces the inaccurate language used by students so that they can appropriate the accurate mathematical term.

5.2. The Case of EML 2008

Preview

In the EML 2008, the blue and green rectangle problem is introduced on the second session of Day 3. After posting the blue and green problem on the board, the teacher explicitly clarifies the intended whole in the problem statement: once by the teacher, once by the student, and once again by the teacher. After a 14-mintue partner work, the whole-group discussion begins with discussing the blue triangle. For the blue triangle, four pairs of students—Britney and Adele; Melody and Ingrid; Bertwin and Kale; and Karl and Calder—present their solutions in a public space. The methods of dissecting and partitioning vary, but all of them name the blue triangle as 1/8. Several students name the green rectangle as 1/8 when explaining the blue triangle, but no separate discussion is made for the green rectangle on Day 3.

On the next day, Day 4, the teacher revisits the blue and green rectangle problem. Instead of drawing lines, the teacher introduces cutouts of the blue triangles and cutouts of the green rectangles and supports students in proving that it takes eight blue triangles (or green rectangles) to cover the whole so one blue triangle (or one green rectangle) is named as 1/8. After proving that the blue triangle is 1/8 and the green rectangle is 1/8 by using the cutouts, the teacher introduces the common errors (1/2 and 1/4) by taking a different whole from the intended whole in the problem statement. The extensive detailed analysis of 32-minute instructional interactions of Day 3 and 15-minute instructional interactions of Day 4 managed by the teacher, Ms. Ball, for teaching the blue and green rectangle problem in the EML 2008 is provided below.

Extensive Detailed Analysis

The blue and green rectangle problem is introduced on the second session of Day 3 in the EML 2008. The teacher briefly reviews what needs to be done before naming a fraction. Kale brings up the idea that they need to see if it is (divided into) equal sizes and the teacher further probes what needs to be done if it is not (divided into) equal sizes. After reminding that drawing a line helps students see equal parts, the teacher posts the blue and green rectangle problem on the board. In introducing the blue and green rectangle problem, the teacher traces the border of the big rectangle with her finger to clarify what “the big rectangle” refers to. Unlike the EML 2007 wherein students generated problem statements, the problem statements are written on the poster. In the poster, one triangle in the upper left corner of the big rectangle is shaded blue and one rectangle in the lower right corner of the big rectangle is shaded green.

What fraction of the big rectangle is blue?
What fraction of the big rectangle is green?

The teacher asks Manoel to read aloud the problem statements written on the poster and then invites him to come up to the board to check if he understands what the big rectangle refers to. Manoel first points to the blue triangle and then points to the green rectangle, but correctly traces the boarder of the big rectangle with the teacher’s support. After Manoel returns to his seat, the teacher introduces materials that the students can use during their partner work. In introducing materials, the teacher hints that cutting, measuring, laying shapes, or drawing lines could help them solve the problem. Unlike in the EML 2007, the teacher clarifies what the big rectangle refers to and introduces mathematical resources that students could use to solve the problem.

After a brief review about one key idea for naming a fraction (making equal parts), the teacher gives time for students to work on the blue and green rectangle problem with a partner. During partner work, the teacher engages in the following mathematical work.

First, the reference of the big rectangle is solidified. Despite the clarification about the whole at the beginning of lesson, several students express their uncertainties about the whole or incorrectly name the blue triangle as 1/2 by taking a part of the whole. When students name 1/2, the teacher probes the reference of 1/2 and extends its reference to the whole.

Second, several issues that hinder students’ access to the key ideas of the blue and green rectangle problem are resolved. For example, Kale initially names the blue triangle as 2/6 by taking unequally partitioned shapes, counting the blue triangle and the green rectangle together, and excluding the green rectangle from the whole. Through engaging in the extensive conversation, the teacher provides supports for Kale to access the key ideas of the blue and green rectangle problem.

Third, the teacher encourages students to explain how the specific method they used supports their explanation. Being influenced by the method used to prove that the parts are equal in the brown rectangle problem, several students count the little grids inside each shape. For example, Ander and Alexico explain that the blue triangle and the green rectangle are the same because each has 30 squares in it. Counting little squares is one way of proving that each part has equal area but makes it difficult to make a direct connection to why the blue triangle or the green rectangle is called 1/8.

Fourth, the teacher uses the partner to convince each other, especially when they come up with different answers. For example, Bertwin names the blue triangle as 1/2 and Kale names the blue triangle as 1/8. As another example, Ander counts six parts but Alexico counts eight parts. Not dismissing this moment, the teacher takes this as an opportunity for Kale and for Alexico to convince their partners.

Lastly, the teacher does not just compliment the correct answer that students produced but further challenges students to defend their correct answer. Getting the confirmation that the blue triangle is 1/8 and the green rectangle is 1/8, the teacher further challenges students why they are both 1/8 despite the different shapes.

After a 14-minute partner work, the teacher convenes the class to begin a whole-group discussion. She first shares her observation that some students changed their minds but many students have a similar answer for the problem. In reviewing students' notebooks, 13 out of 26 students clearly recorded 1/8, two students wrote "half of a quarter", and one student wrote "1/2 of 1/4" in their notebooks. Karl, Calder's partner, wrote "1/2+ 1/4 =1/8" in his notebook. Konrad wrote 1/2 and Melika wrote four different fractions (1/8 or 2/4 or 1/4 or 1/2). Karina expressed that she did not know fractions, Jacey completely crossed out her original answer, and five students did not have a record of answers[21].

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[21] Because the teacher distributed another enlarged drawing to students, some of them only drew, colored, or wrote in the separate sheet, but did not make records in their notebooks.

Table 5.3. Answers that the EML 2008 students wrote for the blue and green rectangle problem in their notebooks

Answers		The number of students
Same answer for each question	1/8	13 (Alexico, Ander, Chantal, Dalton, Ingrid, Jamila, Jack, Linda, Marcella, Manoel, Melody, Molly, and Saniya)
	Half of a quarter	2 (Adele, Britney)
	1/2 of 1/4	1 (Calder)
	1/2 + 1/4 =1/8	1 (Karl)
	1/2	1 (Konrad)
	1/8, 2/4, 1/4, 1/2	1 (Melika)
Different answer for each question		0
Unrecognizable		1 (Jacey)
No records or absent		6 (Bertwin, Delilah, Galvin, Kale, Melinda, Karina)

At the beginning of a whole-group discussion, the teacher announces that they will share different methods used for solving the problem and they will discuss how to prove it. The teacher traces the big rectangle with her finger once again—as a fourth time identification about the whole publicly—and then give a turn to Karl and Calder to propose their answer for a fraction that is shaded in blue.

Teacher: Can somebody say what you and your partner decided about the fraction of the whole rectangle (tracing the big rectangle with her hand) that the blue is? What did you and your partner decide? Karl and Calder?
Karl: Uhm... That are…
Teacher: What did you-
Karl: -it's a-
Teacher: -what fraction-
Karl: -it's equal parts.
Teacher: Yeah, of the blue
Karl: Oh, it's a uhm-
Teacher: What fraction of the whole is the blue?
Karl: Yeah, it's one-one and a half.
Teacher: One what?
Karl: One-(to Calder) wait, what is it?
Teacher: What do you have written down?
Calder: Well we thought that it was... uhm, just that square would be one-half, but if you were measuring the whole thing, I thought it would probably be something like one-half out of four.

Teacher:	One-half out of four. Let's write that one down for a minute. I think you told me when I walked up to you-you said one-half of a fourth.
Calder:	Yes.
Teacher:	Right? So, I'm gonna write that down and keep it on the board. (1/2 of 1/4) Okay?

After several overlapping talks between Karl and the teacher, Karl explains that it is "equal parts" but does not nominate a particular fraction. The teacher asks again about the fraction and Karl says with a stutter, "One-one and a half." Because of stuttering, it is not clear whether Karl intends to nominate "one and a half" or "one-half." As the teacher asks again what Karl said, Karl whispers to his partner, Calder, and Calder supplements Karl's explanation. Calder first identifies one-half for that rectangle (the smaller rectangle in the upper left corner of the big rectangle) and then nominates one-half out of four for the whole rectangle. Karl initially provides an idea of equal parts—even though he does not provide a clear idea of whether it is one and a half or one-half—and Calder provides a fraction identifying the whole that he uses as a reference. Even though he chooses inaccurate language ("that square" rather than "the smaller rectangle"), misses a process of getting the fraction (how to get one-half), and treats the parts as separate entities ("one-half out of four" instead of "one-half of one-fourth"), he accurately lays out the key idea of taking the whole accurately while implicitly refuting one-half as a candidate answer. The teacher refers to her interaction with Calder during the partner work and then revises "one-half out of four" to "one-half of a fourth" based on her observation. Using her observation, the teacher provides support for Calder to translate his written arithmetic expression (1/2 of 1/4) to verbal expression (one-half of a fourth).

After writing "1/2 of 1/4" on the board, the teacher elicits another answer for the blue triangle without further pursuing its conversion into a single fractional amount. Bertwin and Kale, who have been vigorously raising their hands to present their proposal from the beginning of lesson, get a turn. Bertwin proposes 1/8 as an answer for the blue triangle. Instead of further probing how he got 1/8, the teacher asks students whether someone thinks about another answer but no other proposals are made. The teacher shares her observation that a lot of people thought that the answer is 1/8 but that they used different ways to prove it. There are only two candidates nominated, 1/2 of 1/4 and

1/8, which are actually the same answer. At this point, the teacher turns students' attention to the method of proving the answer. Britney comes back to Calder's idea and expands on the idea of how to get the answer.

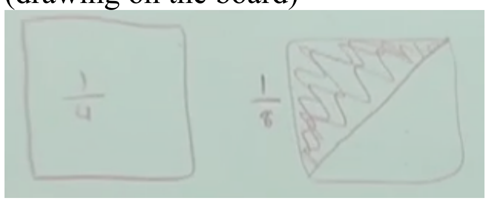
Britney: Yeah. Well I just named it differently. I did- I called it half of a quarter.

Teacher: Just a second. Can I have no side conversations please, okay? Talk a little louder, Britney.

Britney: I named it- instead of one-eighth, I said half of a quarter.

Teacher: Okay. Could you show us what you did?

Britney: (drawing on the board)



Teacher: Okay, so tell us what you're thinking. Okay, so explain to the class, Britney. Or Adele. I don't know which one of you is doing the talking right now?

Britney: Well this (pointing to 1/4 in her drawing) is one square out of the… one square out of the whole entire thing would be one-fourth, and it's half of that square, like this one (pointing to the blue triangle on the poster), would be one-eighth.

Teacher: Okay. How do you know these are equal areas?

Britney: Because they-uhm-how do I know that this (blue triangle) is equal to this (the white triangle next to the blue triangle)? Because, uhm, it is divided directly into half and you can tell, because… because… because if you count the squares, then they're gonna be the same.

Teacher: Okay. Any comment about what Britney and Adele are showing? Okay. So, you're saying there are four equal parts and then each of those equal parts is divided into half? And you're calling one of those eighths? How did you decide that eight was a number of the part of the total?

Britney: Because there are four wholes. If this was (pointing to the green rectangle), there was no line, then they would each be, each will be one-fourth, and then if you cut it in half, if you cut all of them in half like,-like they are cut in half, then you'd get eighths because it's-you take denominator and multiplied it by two.

Teacher: Okay. Let's get someone, good job girls. Let's have somebody who has a way they can take what Britney and Adele said, do you have a way of showing the class that you can show the eight equal parts? I know some people have methods that they thought that they used that showed the eight equal parts. Melody, do you want to show yours? Were you working with Ingrid?

Britney positions that she names it differently and then explains the process of getting a half of a quarter. In responding to the teacher's request to show what she did, Britney draws two rectangles on the board. She first draws one square (it is actually a rectangle instead of a square in the problem), which is isolated from the big rectangle and names it as 1/4. Next, she draws another rectangle with diagonal line, shades one triangle, and names it as 1/8. Different from Calder, who starts from the blue triangle and its nearest circumscribed rectangle and then expands to the big rectangle, Britney starts from the big rectangle and narrows her reference (explaining how she got one-fourth and then how she got one-half out of one-fourth). Not only does she provide the incorrect name for the figure ("one square" instead of "one rectangle"), but she also does not demonstrate how each of the four rectangles out of the big rectangle is named one-fourth and how each of the two triangles out of the small rectangle is named one-half. Different from her initial position that she names it differently, she concludes that it is 1/8 after going through all of the processes. The teacher catches the missing part of Britney's explanation and asks about the reason for equal areas. Britney explains that the blue triangle and the white triangle are the same if counting the little squares inside of each of two triangles. In the previous lesson of the brown rectangle problem, counting the little squares in the grids is used as evidence to prove that each piece has an equal area. In naming a fraction that is shaded in blue, it is not easy to count little squares inside of the triangle because it also includes half of the little squares. However, the teacher does not challenge further the method of counting the little squares in the grid but moves on by asking for comment. As no one comments, the teacher summarizes Britney's explanation by adding the information of "equal parts" and challenges how she gets eight parts. Britney once again goes through the process and explains how she gets 1/4. However, at this point, she names "four wholes" instead of "four parts" and proposes the algebraic method. Britney's method provides access to the fraction concept, a part of a part, which arithmetically leads to the multiplication of two fractions (both multiplier and multiplicand are fractions). The solution that Britney suggests at the beginning (1/2 of 1/4) might be accessible for students, but the final stage of getting 1/8 might not be easily accessible for other students without understanding about the arithmetical approach (which is not an established knowledge). Based on what Britney presented, the teacher asks other students to show eight equal parts, instead of delving into how to get 1/8 from 1/2 of 1/4 arithmetically.

Melody:	Me and Ingrid took and drew a line to go up to the (gesturing to draw a vertical line on the upper right side of the rectangle)-
Teacher:	Do you wanna do it? You can do it.
Molody:	(draws lines on the poster with help of Ingrid)
	
Teacher:	How many other people drew lines into the drawing? Raise your hands if you and your partner drew some line.
Students:	(several students raise their hands)
Teacher:	How many people used scissors to cut pieces?
Students:	(several students raise their hands)
Teacher:	Okay. Can everyone look at what Melody and Ingrid are showing, please? Speak up, girls. What did you do?
Melody:	Then we counted all the parts and we counted eight parts, so for blue, we figured out that that was one-eighth shaded and the green was one-eighth shaded too.
Teacher:	Okay, so there are-you have different kinds of shapes in there. Could you explain how you know those are eight equal parts?
Melody:	Because there're four squares, and each square has two parts in it.
Teacher:	Are the two parts equal or not equal? Remember equal parts is-
Melody:	-Equal.
Teacher:	How can you show that they're equal? Can you just talk us through that? For each square, can you show us how you know that they're equally divided inside that square?
Melody:	If you count the squares in them, they'll have all equal squares in them (counting the little grid inside of each part).
Teacher:	So just take your fingers and show us where the four equal parts are. You don't have to count them. Just show us which of the four squares you're looking at, Melody?
Melody:	We're looking at this one, this one, this one, and this one (pointing to each of four small rectangles).

While Melody draws lines on the poster, the teacher surveys how many students draw lines and how many students cut into pieces. After drawing lines, Melody continues her explanation that she counts eights parts and one of them is shaded in blue and one of them is shaded in green. Melody misses the language of "equal" in her explanation and the reason for drawing lines in her explanation. Instead of pointing out that Melody misses the information of "equal" in her explanation, the teacher challenges her by pointing out how two different shapes (four triangles and four rectangles) make up eight equal parts. Not embarrassed with the teacher's continuous challenges, Melody explains that each of four squares has two equal parts. The teacher continues to challenge Melody's explanation by asking for the evidence of equal division. As Melody starts to count the little grids on the poster, like Britney—method is used to prove that each part has the same size for the brown rectangle problem in the previous day—the teacher asks Melody to trace four equal parts she identified instead of counting the little grids on the poster. The teacher does not ask to repeat or restate one's explanation up to this point, but asks for a collective agreement on whether each of these four small rectangles is equally divided into half, which produces eight equal parts.

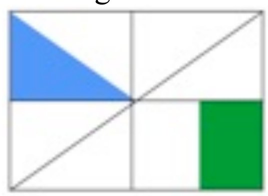
Teacher:	Okay, do people agree with Melody and Ingrid that those squares are each divided in half exactly into equal areas? Do people agree with that or not agree with that?
Students:	Agree.
Teacher:	Yes? I can't really hear you.
Students:	Yes.
Teacher:	You do? Okay. So count us where the eight equal parts. Can you count? One, two, three, and four like that and show us the eight equal parts? (walking to the poster and then starting to count the two triangles on the upper left side of the big rectangle) Like this, one, two, like that? Can you show us eight equal parts?
Melody:	One, two, three, four, five, six, seven, eight.
Teacher:	Mel-Ingrid, do you want to add anything to this?
Ingrid:	(shaking her head)

This is the first segment in which the teacher directly asks for agreement or disagreement. In response to the teacher's request for agreement or disagreement, the students show their agreement with Melody's explanation. This means that no one challenges about why the blue triangle and the green rectangle are equal size. The teacher asks Melody to count eight equal parts and then checks whether her partner, Ingrid, has anything to add to Melody's explanation. The teacher checks whether anyone has a comment on Melody's explanation and moves to elicit another explanation from other students. Bertwin and Kale, who initially proposed 1/8, get a turn to explain their answer.

Bertwin: We put a line across uhm… (gesturing to draw a diagonal inside of the big rectangle)

Kale: No.

Teacher: Do you wanna show your piece of paper where your lines are? Maybe that would be easier. Can everyone look up please? Okay, show what you drew please. Hold it up, so the class can see. Now explain that. Maybe, Kale, can you explain what Bertwin is showing-he is holding up?



Kale: Yeah, we, uhm, we uhm, (to Bertwin) hold that.

Teacher: You hold it and you talk. That will work fine. Go ahead.

Kale: We uhm, we counted, we counted uhm, we counted uhm, we counted, all of these (points two triangles on the upper left side of the rectangle), all of these (points to the two triangles on the upper right side of the rectangle and points to the two triangles on the lower left side of the rectangle)… (giggles)

Teacher: You counted all the pieces and what?

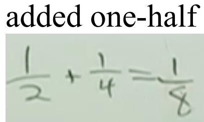
Bertwin: All these shapes

Kale: We count all of these shapes and they was equal and then they came to one, uhm, one-eighth.

Teacher: Okay. So they were all in equal areas and you counted the eighth. That's kind of similar to what, I think what Melody and Ingrid showed too, but you can see it on the paper. Nice job, boys.

After stuttering a little bit, Kale points to the parts that he counted. As the teacher asks what they did after counting all the pieces, Kale adds the information that all of the shapes are equal. The teacher summarizes their explanation and then comments that the method is quite similar to Melody and Ingrid's method. The teacher then gives a turn to Karl and Calder, who proposed 1/2 of 1/4 at the beginning of the whole-group discussion.

Karl: (writing 1/2 + 1/4 = 1/8 on the board) Well, what we did was, we added one-half and one-fourth. And it <u>*became*</u> one-eighth.



Teacher: So what are you-could you say that one more time? I apologize.

Karl: Okay, we did one-half, it <u>*became*</u> one-fourth, it <u>*became*</u> one-eighth.

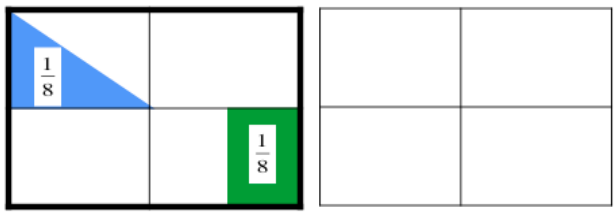
Teacher: But, what do you mean by "it became"?

Calder: Like we added the numbers (pointing to 1/2 and 1/4) and they like-wait, what did-

Karl:	-it's a kind-
Teacher:	I don't think that you were adding them. I think that you were-look at the drawing. I think he said one-half of one-fourth, right? Of one-fourth, not adding them.
Karl:	Yes, that's what we did.
Teacher:	So, can you show how half of one-fourth is one-eighth over there? That's good. Just show it on the drawing. You don't have to write anything at all. How did you get one-half of one-fourth would be one-eighth?
Karl:	How did we? Well…
Calder:	We didn't really use the drawing
Karl:	We didn't really use the drawing…

Calder, who initially proposed 1/2 of 1/4, has a turn to explain his answer with his partner, Karl. At this time, Karl explains that they add up 1/2 and 1/4 and writes "1/2+1/4=1/8" on the board. Instead of explaining how he got one-half and one-fourth and what each of them means, Karl explains that they added one-half and one-fourth and it "became" one-eighth. In responding to the teacher's request to explain once again, Karl maintains his explanation but substitutes the language of "add" with "became" in his repeated explanation. Through his partner work with Calder, along with comments they received from the teacher during their partner work, Karl identifies 1/2 and 1/4 and his partner suggests 1/2 of 1/4. However, in the process of translating "1/2 of 1/4" to arithmetic expression, Karl chooses the operation of "addition" instead of "multiplication" and expresses the process of taking a part from the whole as "became." The teacher probes the meaning of "became" and Calder repeats that they added two fractions. The teacher corrects that it is not adding but it is 1/2 of 1/4. However, the teacher does not correct addition to multiplication. Keeping a focus on "half of one-fourth," the teacher asks Calder and Karl to show how 1/2 of 1/4 would be 1/8 in the drawing. Both Karl and Calder address that they do not use the drawing. At this moment, the second session of Day 3 is almost over, so the teacher suggests working on this problem again the following day.

On the next day (Day 4), the teacher revisits the blue and green rectangle problem and reviews what the class worked on yesterday. The teacher first traces the big rectangle with her finger, draws a line with a marker so that students can easily to identify the whole, and then emphasizes that they are working on the big rectangle. After the teacher briefly explains that a square is a special kind of rectangle, she asks students what to call a square. Ingrid explains the difference between a square and a rectangle. In the previous lesson, some students incorrectly used a “square” to refer to the diagram, but the teacher did not weigh on pointing out misusing the term “square” every time. Instead, when repeating students’ explanation, the teacher replaced “square” with the “rectangle” in the previous lesson and then briefly commented that it is a rectangle not a square at the beginning of this lesson.

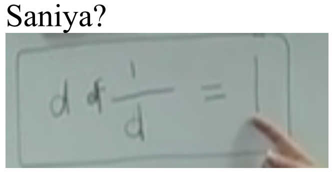


After clarifying the language (“rectangle” not a “square”), the teacher reviews what they decided about what fraction of the whole big rectangle is blue and what fraction of the whole big rectangle is green. Delilah answers that the blue triangle is 1/8 and Marcella answers that the green rectangle is 1/8. The teacher introduces a blank rectangle that is divided into four equal parts and a bunch of cutouts of blue triangles and then asks students to prove how the blue triangle is one-eighth of the big rectangle. Unlike the EML 2007 cohort who drew the diagonal over the green rectangle to name a blue triangle and drew the vertical line over the blue rectangle to name a green rectangle in a whole-group discussion to make all parts equal shapes, the EML 2008 cohort does not make all parts into equal shapes. Using the cutouts allows the students to prove that it needs eight equal triangles to cover the whole.

Marcella: You could put them like all over each square and then count how many there are.

Teacher: Okay. So she said, Marcella says we can put them up-cover it with these, and then see how many there are. If we were right that it’s one-eighth, how many of these should I take? How many of these should it take-if that’s-if we’re right about this, that this (pointing to the blue triangle) is one-eighth of the big rectangle, how many of these should we have to use to cover the whole thing? Raise your hand if you know how many of these should take? Will it take three or four or seven? What will it take to cover the whole thing if that really is one-eighth?

Following Marcella's idea, the teacher specifies her question into, "How many of triangles should it take to cover the whole?" Melinda suggests one. After the teacher covers it up with one triangle, Melinda sees that it is not completely covered yet and revises her answer to four. Still seeing that the whole rectangle is not completely covered yet, Melinda suggests using four more triangles to cover the whole. After covering the whole with eight triangles, the teacher asks whether this proves that one of the whole is called 1/8. At this moment, Karl reminds the teacher of his and Calder's work on the board, but the teacher keeps her focus on proving whether this method proves that one of the triangles is called 1/8.

Teacher:	Does this prove that one of them should be called one-eighth? Can someone explain why that would be? It takes eight of those shapes to cover the whole thing. Why would one of them be called one-eighth? Delilah?
Delilah:	Because it's one shaded in eight parts all together.
Teacher:	Eight, what kind of parts?
Delilah:	Equal.
Teacher:	Eight equal parts. And so we call one of them?
Delilah:	One-eighth.
Teacher:	One-eighth. So, that's just exactly what's on the board. Can everybody look back over at what we just did? What does this say? (pointing to "d of 1/d =1" on the board) Can someone read this with the eighth? How does that show what Melinda and I just did? Saniya? 
Saniya:	Eight of one-eighth equals one whole.
Teacher:	Right. Does everyone see over here, eight of one-eighth equals one whole?
Students:	Yes.

The teacher points to "d of 1/d=1" on the board and gets to the idea that eight of one-eighth make a whole. In a similar manner, the teacher asks how many green rectangles should it take to cover the whole and Kale put eight green rectangles to cover the whole.

Teacher:	So, what if we did it with the green rectangles? Cause we said the green rectangles were also one-eighth. How many of these should it take to cover that rectangle there? Kale? How many of these should it take to cover all?
Kale:	Eight.
Teacher:	What?
Kale:	Eight.
Teacher:	Okay. So, let's exchange those. You wanna come and help me? So, now we're gonna take the triangles off. Okay? How many of these do you think it's gonna take?
Students:	Eight.
Teacher:	Here, I'll take them off and you get to put them on. That's more fun. (giving the green rectangles to Kale) So Kale is gonna try to cover the whole rectangle with the rec—green ones and see if it comes out to be eight.

Teacher:	So, Kale, can you turn and tell the class what that shows?
Kale:	Uhm, eight of one-eighth equals a whole.
Teacher:	Okay, so does that prove that the green rectangle is also one-eighth of the whole?
Students:	Yeah, that proved.
Teacher:	Does everybody see that?
Students:	Yeah.

In a similar manner, Kale shows that it takes eight green rectangles to cover the big rectangle. By showing that 8 of 1/8 equals a whole, Kale proves that the green rectangle is 1/8 of the whole. At this moment, the teacher asks a tricky question to students.

Teacher:	I have a question for you. So, what if somebody came into our class, who hadn't been here before, and they looked at our work and they said "you guys are all totally wrong. You don't get fractions at all." That blue triangles right there… this one… they don't like our answer at all and they say "you're really wrong, that's actually one-half." Is there any way that that could really be right? Is there anything you could imagine that they were thinking that we would make that really right that they think it's one-half? Would you be able to figure out what they were thinking? Why would somebody come into our room and say "You guys are crazy. That's really one-half, not one-eighth." Manoel?
Manoel:	Uhm, because he's just, uhm, the person is just thinking, uhm, that in one of rectangles, not out of the whole entire rectangle.

Teacher:	So what do you think he's looking at?
Manoel:	Uhm, just-
Teacher:	When you say one of-
Manoel:	-Just that area where the rectangle, where the triangle
Teacher:	Just this right here?
Manoel:	Where the triangle is in.
Teacher:	(drawing a line around the left-hand side small rectangle)
Manoel:	Yeah, just that one.
Teacher:	Okay. Do people agree with Manoel? If the person was just looking at this rectangle up here, would they be right that that was one-half?
Students:	Yes.
Teacher:	Can someone say why that would be right? Britney, why would that be right?
Britney:	Because it takes two of those triangles to make one.
Teacher:	Cause it takes two of those triangles right there to make that whole rectangle, so that person could be right if they were paying attention to that being the whole. (putting two triangles)

The class agrees that the blue triangle is 1/8 of the big rectangle and the green rectangle is 1/8 of the big rectangle. Even after the proof, the teacher challenges the students by asking what if somebody calls it 1/2 instead of 1/8, and asks students to think about the reason to get the answer of 1/2. Manoel provides quite a clear explanation that that person who calls the blue rectangle as 1/2 only considers one of the rectangles instead of the whole entire rectangle. The teacher then asks whether other students agree with Manoel's explanation about taking a different whole and gives a turn to Britney to elaborate on the explanation about the number of parts that composes the whole.

Teacher:	What if somebody else came into the room and said "I don't agree with you either, it's not one-half, it's also not one-eighth, it's one-fourth." Is there any way that person could be thinking that this is one-fourth? There's something they could be looking at that could make that the right answer? Is there something that people could be looking-a person could be looking at that would lead them to say that this blue triangle is one-fourth? I'm looking for some more hands. Who else thinks they know? Melika, what do you think?

Melika: Uhm, I think they're just looking, uhm, they're looking at both of the boxes with the, uhm-

Teacher: Come up and show what you're looking at. Watch at Melika shows. She thinks she knows what the person could be looking at.

Melika: I think they're just looking at this (pointing at the lower-bottom rectangle) and this (pointing at the upper-bottom rectangle)

Teacher: Can you show why they could call that one-fourth then?

Melika: Because it takes two, uhm

Teacher: That's right

Melika: two triangles to fill up (pointing to the lower left side of the rectangle), uhm, all four of the… the…things.

Teacher: I think you mean it takes four triangles to fill up the two

Melika: Yeah.

Teacher: So if it takes four triangles, we can hold them up there, would they be right in calling that one-fourth? What would they be calling the whole here? Can you say that again Melika?

Melika: One-fourth.

Teacher: Can you show them? So how could they call that one-fourth, Melika? Can you show them?

Melika: Because this one is shaded and it's four of them…

Teacher: What kind of parts?

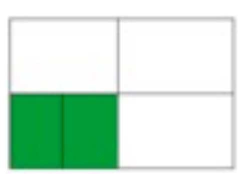
Melika: The… rec… the blue part of the rectangle

Teacher: That's right. The blue part is one out of how many equal parts?

Melika: Four.

Teacher: Four equal parts.

In a similar vein, the teacher asks students to think about any reason why someone calls the blue triangle is 1/4. Melika provides a clear explanation by identifying the whole that someone takes and shows that four of triangles make up the whole. The teacher then shows the following diagram to students to think about what fraction of the big rectangle is covered up.



Teacher:	Okay, who want to say what number they wrote down? What fraction did you write for how much of the whole rectangle is covered in green right now? Saniya?
Saniya:	Two-eighth.
Teacher:	Two-eighth. Why did you pick two-eighths?
Saniya:	Because it takes eight rectangles to cover the whole thing and it's only two covering.
Teacher:	Okay, that was a superb explanation. Could you say that one more time really loudly? I want everyone look at Saniya. That was excellent. So you said it's two-eighth. Say again how you figured that out.
Saniya:	Because it takes eight of those, uhm, green rectangles to cover the whole thing and only two of them is on the rectangle.
Teacher:	Could everyone hear what she said?
Students:	Yeah
Teacher:	It takes eight rectangles to cover this (tracing the whole) and they're all equal. Did you say they are equal? Maybe I'm adding that. They're equal and two of them are covering the shape. Does someone have a different answer beside Saniya? There's another answer you could give but that was very good, Saniya. What's another number you could give what part is shaded? Dalton?
Dalton:	One-fourth.
Teacher:	How could you say one-fourth?
Dalton:	It could be one-fourth because there's four areas and one shaded.
Teacher:	There's four equal areas here and one of them is filled in, so you could have written one-fourth also. How many people wrote one-fourth in your notebook? Raise your hand if you wrote one-fourth.
Students:	(a couple of students raise their hands)
Teacher:	How many people wrote two-eighths?
Students:	(more than half of students raise their hands)
Teacher:	Okay so both of those numbers are right. They are two different ways to write it. This one has to do with making eight equal parts and this one is about making four equal parts.

The example the teacher chooses is different from the original blue and green rectangle problem because more than one part is shaded. Using the ideas that the class established, Saniya provides an explanation for why it is called 2/8 and Dalton provides an explanation for why it is called as 1/4. The first session of Day 4 is wrapped up with students writing in their notebooks about one thing they learned about fractions.

Summary

The EML 2008 students propose two answers (1/2 of 1/4; 1/8) for the blue triangle and propose one answer (1/8) for the green rectangle. Only correct answers are proposed by the students, but the numerical representations, the way of dissecting, and the geometrical representations vary. The proportion of incorrect answers produced by not taking the intended whole (e.g., 1/2) is pretty low. The presented methods do not dissect the whole into the same shapes, but the issue of dissecting into different shapes is not raised by the students. The method of counting the little grids is adopted by several students to prove that each part has the same size. After the collective agreement about the answer, the teacher introduces two common incorrect answers (1/2; 1/4) and encourages the students to think about why someone might come up with those answers. In explaining the blue and green rectangle problem on Day 3, the EML 2008 students initially use the language incorrectly, inaccurately, and inconsistently; grant incorrect geometric names; does not use the comparative language in describing different sizes of the rectangle; does not build a connection between the numerical representation and the pictorial representation; and does not equip sufficient mathematical knowledge to handle "a fraction of a fraction" either algebraically or geometrically. Even at the beginning of the whole-group discussion, the EML 2008 students have clear understanding about the intended whole. The students tried to prove the equal parts by counting the little grid inside of the shape, which is influenced by proving equal parts for the brown rectangle problem at the beginning of Day 3, but become fluent in proving the equal parts by explaining how many equal parts cover the whole.

To support students' development of mathematical explanation for the blue and green rectangle problem, the teacher clarifies the intended whole while launching the task, remediates some errors which interrupt the access to the key idea of the blue and green rectangle problem during individual work, provides mathematical supports upon an individual student's request but does not address the issues in a public space; does not jump into the mathematical knowledge that the class does not establish yet (e.g., 1/2 of 1/4); invites the students to the board, allowing them to clarify what is being explained and to build a connection between a verbal explanation and a pictorial representation; uses the established knowledge (d of 1/d = 1) to support students to explain; uses the cutouts to support the students resolving the issue of dissecting the whole into difficult shapes, to prove that each part has equal size, and to identify the whole; does not diverge into the issue of why "1/2 + 1/4 = 1/8" is not a correct mathematical statement; and replaces inaccurate language and fills in the missing word so that the students can appropriate the accurate mathematical term.

5.4. The Case of EML 2009

Preview

In the EML 2009, the blue and green rectangle problem is introduced on the first session of Day 5. After the students read the problem statement, the teacher traces the big rectangle on the board once but does not extensively investigate what the big rectangle refers to at this point. Due to the time constraint, the students are mostly engaged in working on the blue and green rectangle problem either individually or with a partner on Day 5. On the next day, Day 6, the teacher revisits the blue and green rectangle problem. The teacher shares her observation that most students have the same answer for the blue triangle and for the green rectangle and then addresses the need for proving whether it is the right answer. Before students come up to the board to explain their answers, the teacher introduces resources that the students could use to prove: (1) the working definition of fraction; and (2) blue triangles cutouts and green rectangle cutouts. For the green rectangle, Aiyana proves 1/8 by showing that it takes eight green rectangle cutouts to cover the whole. For the blue triangle, Manley and Malik prove 1/8 by showing that it takes eight blue triangle cutouts to cover the whole. The teacher further challenges how the two different shapes are the same. The extensive detailed analysis of 24 minutes of instructional interactions of Day 5 and 51 minutes of instructional interactions of Day 6 managed by the teacher, Ms. Ball, for teaching the blue and green rectangle problem in the EML 2009 is provided below.

Extensive Detailed Analysis

The blue and green rectangle problem is introduced on the first session of Day 5 in the EML 2009. The teacher introduces the blue and green rectangle problem by framing that it helps students get a better working definition of fraction. The teacher puts the poster on the board, in which the triangle on the upper left corner of the big rectangle is shaded blue and the rectangle on the lower right corner of the big rectangle is shaded green. The teacher asks students to read aloud the problem statement and then traces the big rectangle with her finger but does not extensively investigate what the big rectangle means at this point as she did in the EML 2008.

What fraction of the big rectangle is the blue region?
What fraction of the big rectangle is the green region?

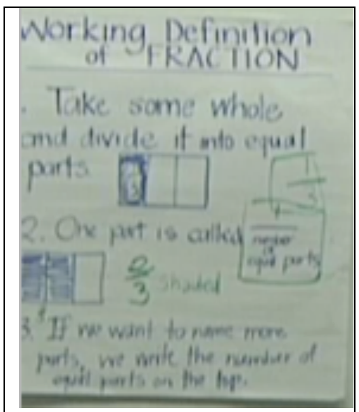
After a 17-minute individual or partner work[22], the teacher announces that she will give a few more minutes for students to work on the problem by themselves because she does not want to rush some students who are still working on the problem. After the teacher gives three additional minutes to work on the problem individually or with a partner, she wraps up the lesson to take a break. In the students' notebooks from Day 5, 12 out of 25 students clearly recorded 1/8 in their notebooks and two students wrote "1/2 of 1/4." Four students recorded the incorrect answers by taking a different whole than the intended whole in the problem statement (one student wrote 1/2 and three students wrote 1/4).

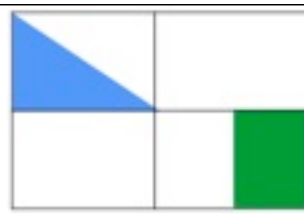
Table 5.4. Answers that the EML 2009 students wrote in their notebooks for the blue and green rectangle problem

Answers		The number of students
Same answer for each question	1/8	12 (Aiyana, Amelia, Callie, Evan, Levi, Malik, Marcellus, Mannis, Natania, Sandra, Tiara, Tonya)
	1/2 of 1/4	2 (Elina, Ricky)
	1/2	1 (Jacqueline)
	1/4	3 (Jana, Manley, Riya)
	2/4	1 (Collin)
	8/4	2 (Akilah and Nina)
	2/6	1 (Alvan)
Different answer for each question		0
Unrecognizable		1 (Marlais)
No records or absent		2 (Dante, Teri)

[22] The teacher does not assign a partner for solving this problem. Depending on his or her own choice, each student works either individually or with a partner.

The blue and green rectangle problem is revisited on the first session of Day 6. The teacher opens a whole-group discussion by sharing her observation that most students answer 1/8 for the first problem ("What fraction of the big rectangle is shaded blue region?") and 1/8 for the second problem ("What fraction of the big rectangle is shaded green region?"). While introducing the answer that many students come up with, the teacher addresses the need for proving if it is the right answer. The teacher then introduces two resources that students can use to prove their answers: (1) the working definition of fractions that the class built together; (2) blue triangles cutouts and green rectangles cutouts. There are three posters displayed on the board: poster with the working definition of fraction, poster with the problem, and poster displaying a number of blue triangles cutouts and green rectangles cutouts.





What fraction of the big rectangle is the blue region?
What fraction of the big rectangle is the green region?

The blue triangle problem is written first on the poster, but the teacher starts to elicit an explanation for the green rectangle problem first. She asks students to take advantage of the cutouts to explain why the green rectangle would be called 1/8 or not. Aiyana comes to the board and covers the big rectangle with eight green rectangles. In the notebook, Alayana wrote:

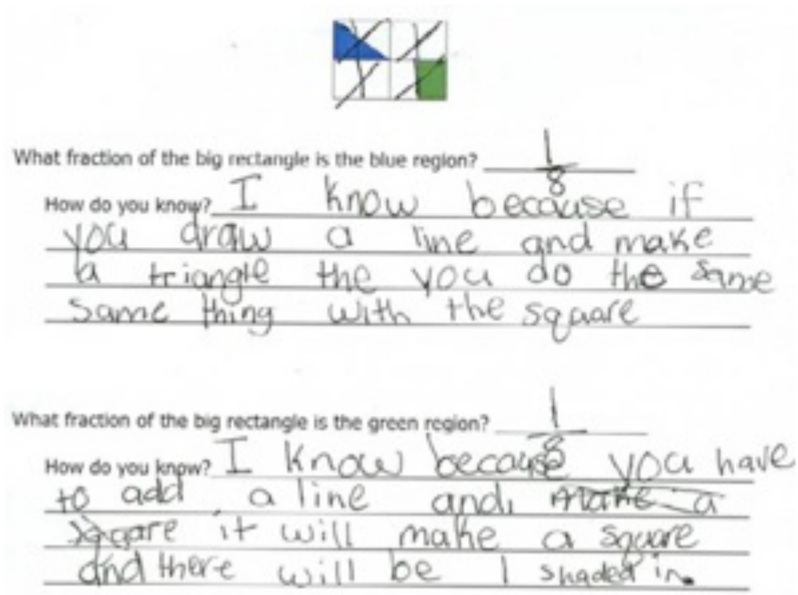


Figure 5.4. Aiyana's answer for the blue and green rectangle problem

Teacher:	Okay, can you now explain what you've done and what that makes-what you think?
Aiyana:	Well, I put all the green squares over the rectangle, even over the blue, because it's asking what fraction of the big rectangle is the green. So you would want to put the greens on it.
Teacher:	Okay, so can you look at the working definition? Aiyana, can you use that to explain the name of the fraction that's one of the greens? What does the first thing say on the working definition?
Aiyana:	Take some whole and divide it into equal parts.
Teacher:	Did you do that?
Aiyana:	Yes.
Teacher:	Okay, she used the green rectangle, to cut—to mark off the rectangle into the equal parts. Can everyone see that?
Students:	Yeah.

In response to the teacher's request, Aiyana explains the process of covering the whole with green rectangle cutouts, even over the blue triangle. Aiyana incorrectly names the green rectangle as the green square, but the teacher does not problematize the incorrect use of term at this point. Aiyana describes the process of covering the whole with green rectangle cutouts, but does not provide an explanation for why it would be called 1/8 or not. Noticing that Aiyana does not utilize the important ideas from the working definition of fraction, the teacher supports Aiyana in improving her explanation by using the working definition of fractions as written on the poster. Aiyana reads the first idea (take some whole and divide it into equal parts) and the teacher checks whether she follows the first idea. At this point, the teacher does not further interrogate what Aiyana calls as a whole but briefly adds that the rectangle is cut into equal parts. Because of using the cutouts, it is evident that they are all equal parts. The teacher supports Aiyana in making up her explanation by paying more attention to the second and third idea of the working definition of fraction.

Teacher:	So what would one part be called in that then? One over the number of equal parts? How many equal parts are there?
Aiyana:	Eight.
Teacher:	Okay, so one of them is called?
Aiyana:	One?
Teacher:	Okay, so let's read number two again. This is important. This is really helpful what Aiyana is doing. So, it says one part is called one over the number of equal parts. So how many equal parts are there, Aiyana? In this?
Aiyana:	Eight
Teacher:	How many?
Aiyana:	Eight.
Teacher:	Eight. So one of them is called? One-eighth. (writing 1/8 on the board) Okay, so can you show us how to see one-eighth up there? How many eighths do you have right now? (tracing the whole rectangle with her fingers)
Aiyana:	Eight
Teacher:	Can you write down that number?
Aiyana:	(writes /8 on the board)
Teacher:	Uh-huh. And how many do you have?
Aiyana:	Eight.
Teacher:	So write eight on the top.
Aiyana:	(writing 8/8 on the board)
Teacher:	Can somebody use the third point on the working definition to explain (clearing her throat), excuse me, why Aiyana can write eight-eighths for all of them? Can somebody use the third thing that's on the working definition to explain why she just wrote eight-eighth for all of them? Malik?
Malik:	Because she covered them all up and made one whole.
Teacher:	What does it say on number three though?
Malik:	If we want to name more parts, we write the number of equal parts on the top.

Teacher: Okay, so how many parts was one of them?
Malik: One of them…
Teacher: One of them is called?
Malik: One-eighth.
Teacher: One-eighth. And how many one-eighth does she have?
Malik: Eight.
Teacher: Can you count the one-eighth for us? Up there? Can you count one-eighth, two-eighths, like that?
Aiyana: 1/8, 2/8, 3/8. 4/8, 5/8, 6/8, 7/8, 8/8.

Through these exchanges, the teacher supports Aiyana in elaborating on why the green rectangle is called 1/8 which includes: (1) there are eight equal parts; (2) one of them is called 1/8; (3) it takes eight of 1/8 to cover the whole; and (4) counts by 1/8 to 8/8. The teacher then asks comments for Aiyana.

Teacher: Okay. Comments for Aiyana? Levi?
Levi: I agree with her. I think it's right. I do think it's right because, uhm, there's no space left if you use eight of them, so that's 8/8 or 1/8 is equal. Well, so there's one green is equal to, well, there's, ugh, I can't
Teacher: One green.
Levi: There's-so there's eight greens in the whole and one of them would be 1/8.
Teacher: Okay. So, Levi just did what I talked about on Friday. When I said "what do you think about so and so as an answer?" you can say you agree or disagree, you can do what he just did. He explained in his own way what he thinks she did. That was a good job, Levi. Aiyana, do you want to say anything else about what you did?
Aiyana: (shaking her head)

In positioning the agreement with Aiyana, Levi explains that eight of green rectangles make up the whole thus one of green rectangles is called 1/8. After Levi's comment, the teacher lets students do some practices problems with naming a fraction using the working definition of fraction on the poster.

Teacher composed;

Akilah answered 6/8.

Evan composed;

Marlais answered 4/8.

Sandra composed;

Elina answered 4/8.

After practicing in naming a fraction with these three examples, the teacher makes a transition to naming a fraction that is shaded blue. Similarly, the teacher establishes the expectation that students could take advantage of the blue triangle cutouts to prove the answer. While Malik and Manley come to the board to cover the whole with blue triangles, the teacher asks other students to explain what they should do with these cutouts and why using cutouts is helpful. Marcellus explains that it is helpful to find out whether they are the same size and Levi explains that it is helpful to figure out how many equal pieces are in a whole.

Manley: We covered just like we did, just like we did with the green squares. We covered them all up with the rectangle.

Teacher: Manley, could you say that again and Malik will talk next? Could you say it while facing up? Cause you were talking to the board. And we're over here.

Manley: We covered them up, just like they did with the green squares. The green rectangles, but except we did with the triangles, the blue triangles.

Teacher: Okay and Malik, do you wanna add what you found out from doing that?

Malik: I found out that…that are… that… since this rectangle already here, I would have imagined this by counting every single one and added that to the fraction.

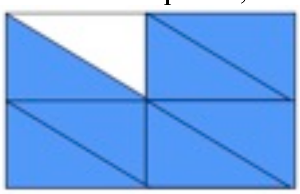
Teacher: What is it called now that rectangle? It's not a rectangle. What did you call it?

Malik: It's, I'm sorry, a triangle.

Teacher:	The big, the big thing is a rectangle, but the little ones are triangle. How many of those blue triangles does it take to cover the rectangle?
Malik:	Eight.
Teacher:	Eight. So what's the name of one rec—one triangle?
Malik:	One-eighth.
Manley:	One-eighth.
Teacher:	Can someone explain why they should call the blue one-eighth from our working definition? Look at the working definition. It says, takes some whole, and then divide it into equal parts. They've already done that. And number two tells you what to name one of the parts. Can someone use that to say what one blue triangle should be called? How about somebody other than Levi? I'd like to see other hands today. Okay? How about Natania? What would you call one blue part if it takes eight of them to make up the whole?
Natania:	One part is called one over the number of equal parts.
Teacher:	And how many equal parts are there?
Natania:	Eight.
Teacher:	Eight. So what's the fraction? What fraction is called-is the blue?
Natania:	One-eighth.
Teacher:	One-eighth. Do people agree with Natania?
Students:	Yeah.
Teacher:	Okay. How many eighths are on the rectangle right now? If one of them is called one-eighth, how many triangles are on the big rectangle right now? Tonya?
Tonya:	Eight.
Teacher:	Eight. So, what's the fraction that we would use to describe how many blue triangles are covering? What fraction of the whole is covered right now?
Tonya:	Eight-Eighths.
Teacher:	What is it again?
Tonya:	Eight-Eighths.
Teacher:	Is Tonya right eight? Eight-eighth of the rectangle is shaded blue right now?
Students:	Yeah.
Teacher:	What does that mean if we say eight-eighths of the rectangle is shaded blue? What does that mean? Aiyana?
Aiyana:	It's all of the rectangle.
Teacher:	It's all of the rectangle. How do you know that eight-eighths is all of it? Is there an easy way to tell that is has to be all of it? Malik?
Malik:	Because it's all shaded in
Teacher:	It's all shaded in. And if you have one-eighth is one of the equal parts, if you take eight of them, it means you've got all equal parts back again. Does that make sense?
Students:	Yes.

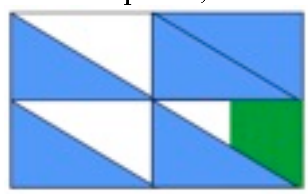
In making a reference to the green rectangle which was just explained, Manley describes his actions. In building a structure of explanation, the teacher supports them in elaborating on their initial explanation by using the ideas on the working definition of fraction. In a similar vein, the teacher lets students do some practices with naming a fraction with the blue triangles.

Teacher composed;



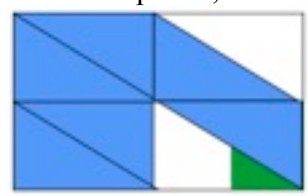
Tiara answered 7/8.

Jana composed;



Aiyana answered 5/8.

Elina composed;



Marlais answered 6/8.

The first part of whole-group discussion mainly focuses on proving the answer of 1/8 by covering the whole with cutouts of each shape (green rectangle and blue triangle) and linking it to the ideas of the working definition of fraction. Even after proving that the green rectangle is 1/8 and the blue triangle is 1/8, the teacher makes students practice naming a fraction for the big rectangle with more than one equal parts shaded. Until this point, other incorrect answers—for example 1/4 or 1/2—are not discussed yet in the EML 2009.

The teacher makes a transition to the second part of discussion. Even after proving that the blue triangle is 1/8 of the big rectangle and the green rectangle is 1/8 of the big rectangle by covering the whole with the unit, the teacher challenges students to think about how they are the same despite the different shapes. The teacher gives the floor to Nina.

Teacher: Okay, so now we have a puzzle that we wanna try to work on. Okay? The puzzle is how could it be that the blue and the green are both called one-eighth? They look totally different. Like, one's a- this is a triangle and this is a rectangle. And we're saying that this is one-eighth of the whole, but we're also saying that this is one-eighth of the whole. How could that be? Do they look same to you? Nina, what do you think?

Nina: I think it's because they are both half of one shape of the whole box.

Teacher: Okay, do you wanna come up and show what you're talking about? I'm gonna take these off. Can people hear what Nina just said? Could somebody repeat what Nina said, while she's getting ready to show it? Somebody beside Malik? Was someone else here listening carefully to what she said?

Students: (silent)

Teacher: You didn't hear because it wasn't loud enough?

Teacher: (To Nina) Could you say it one more time to give more people a chance to hear?

Nina: I think it's because the, the shapes are one half of the whole.

Teacher: Okay, what did she say, Riya?

Riya: She said because she thinks that the shape is half of the whole.

Teacher: Okay, she thinks that the shape is half of the whole box, I think you said earlier. Right?

Nina: (nodding her head)

Teacher: Okay. Can you show us what you are talking about?

Nina: Uhm, this (blue triangle) is half of this whole box (pointing to the upper-left side rectangle)

Teacher: Say that again?

Nina: This, this is, this one (blue triangle) is half of this whole box (pointing to the left-upper side of the big rectangle)

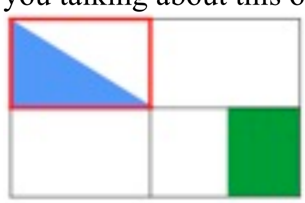
Teacher: Okay, point your finger on the box you are talking about right now.

Nina: This one.

Teacher: Can you put your finger around the edge of it, so we can see what you are talking about?

Nina: (tracing the upper-left side of the big rectangle)

Teacher: Okay. The thing you are talking-can I draw a line around it? Are you talking about this one right here?



Nina: Uh-huh.

Teacher: Can I make a red-like a fence around it? Okay, now try to explain to the class again what you are saying.

Nina: If you put another triangle right here, it will make a whole. And if you take this off, it's still half of this box.

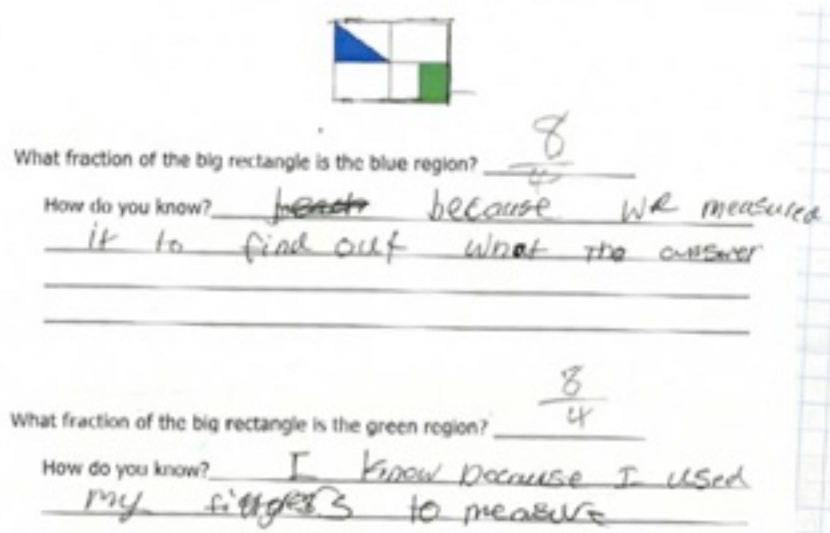


Figure 5.5. Nina’s written explanation in her notebook from Day 5

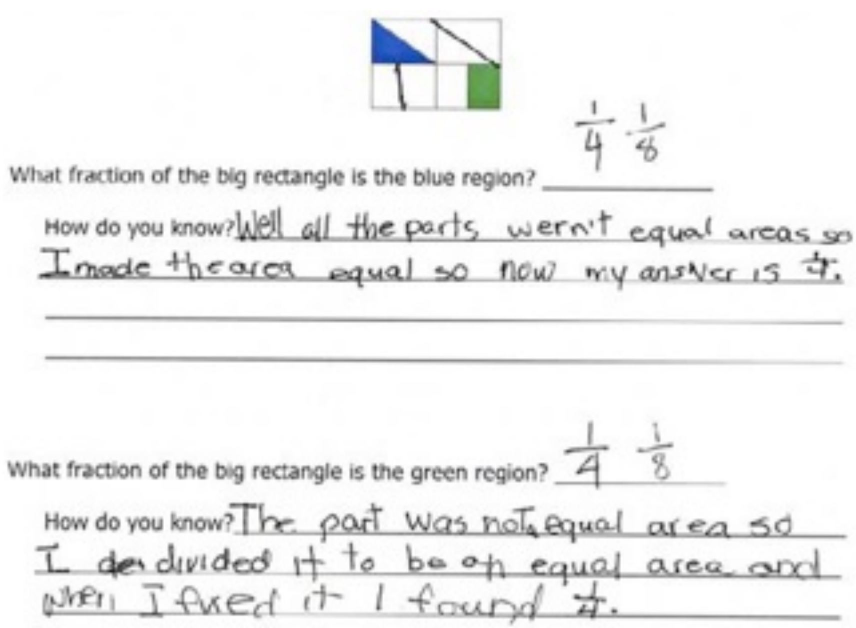


Figure 5.6. Riya’s written explanation in her notebook from Day 5

Nina initially explains that both the blue triangle and the green rectangle are *one half of one shape of the whole box*. While Nina comes to the board to show what she refers to, the teacher asks other students to repeat Nina’s explanation. Seeing that not many students raise their hands to repeat Nina’s explanation, the teacher asks Nina to repeat her initial explanation. In repeating her initial explanation, Nina changes “half of one shape of the whole box” to “one half of the whole.” After Nina repeats her initial explanation, Riya repeats the explanation and the teacher repeats Riya's explanation to check with Nina. Nina calls the upper-left corner of the big rectangles as a whole instead of calling the big rectangle as a whole and explains that the blue triangle is half of the whole box because two triangles make a whole. Nina initially wrote 8/4 as an answer and provided an explanation by describing her action during individual work in the previous day. During a whole-group discussion, even though her language was inconsistent ("half of one shape of the whole box" to "half of the whole"), she not only changes her answer from 8/4 to 1/2 but also provides an explanation with a structure. Riya, who also incorrectly takes the intended whole, explains the whole that Nina used, instead of either backing up Nina's argument or claiming her whole. After Nina's explanation, the teacher pays attention to the whole that Nina uses to name a fraction.

Teacher:	Okay, so what whole is Nina looking at right now? What is she calling the whole? We were calling the rectangle the whole, but Nina's explanation is looking at different whole. What is the whole that Nina is paying attention to? In fractions, it's really important to be clear about what we are calling the whole. What is Nina calling the whole right now? Riya?
Riya:	The area?
Teacher:	But which, what part of that is she calling the whole? Can you show us again, Nina? Which one are you calling the whole right now?
Nina:	(pointing to the rectangle on the upper-left side big rectangle)
Teacher:	Just one of the boxes. Can everyone see? And she's saying that the blue-can someone repeat what she's saying about the blue triangle? Levi?
Levi:	She's saying that the blue triangle is half of one of the-one-fourth of the whole box, like-
Teacher:	Say that again
Levi:	It's one-half, it's on- half of one of boxes. One of the several boxes
Teacher:	One-half of one of the boxes, and you're saying that the boxes are what? You said one-fourth
Levi:	One fourth of the whole.
Teacher:	One-fourth of the whole thing (tracing the whole rectangle with her finger).

The teacher does not evaluate whether Nina names the blue triangle correctly or not. Instead, the teacher supports students to pay attention to the different whole between they used for naming it as 1/8 and Nina used for naming it as 1/2. The teacher asks Nina to point out the whole that she refers to and then asks other students to repeat Nina's explanation. Levi explains that Nina sees the blue triangle as one-half of one-fourth of the whole box. The language of "one half of the shape of the whole box" and "one half of the whole" that Nina initially used is refined by "one-half of one-fourth of the whole box" by Levi. Instead of diverging into how 1/2 of 1/4 equals to 1/8, which is not an established knowledge yet, the teacher goes back to Nina to explain how this idea is helpful to explain that the blue triangle and the green rectangle are the same.

> Teacher: Okay. Now, why, why does it help you think about the green and the blue? Cause you said this would help, when I said (sniffle), "how can those be the same?" you thought this would help.
>
> Nina: Uhm, because if you keep putting two of these on each box, uhm, they are the same as the, this whole thing (pointing to the upper-left side rectangle).
>
> Teacher: Okay, so she's saying if we keep doing that, in each time, we'll have two of these in every box. Do people agree with her? So, now the question is, what is that have to do with the greens? Cause we're also calling the green is one-eighth, why we are calling this the same? Are we calling these-are these really the same area? This (triangle) and that (rectangle)? Could you turn to your partner right now and think about this question. Are the green and the blue, (to Nina) you can sit down, are the green and the blue really the same area? Cause you are calling both one-eighth of the whole, so how would you explain that? If you need to draw something to talk to your partner, you can also draw. See if you and your partner can agree. Are they the same area or not?

Based on the assumption that the small rectangle circumscribing the blue triangle and the small rectangle circumscribing the green rectangle are the same, which is apparent to students, Nina explains that two of the shapes make up each (of four small) box. Because Nina only explains about the blue triangle and its circumscribing rectangle, the teacher asks students how it relates to the green rectangle and asks them to think about how the blue triangle and the green rectangle are the same area. At this point, the teacher gives students one minute to talk with their partners. The teacher resumes the whole group discussion to share an idea of showing that the green rectangle and the blue triangle have the same area. Jana and Riya, who initially recorded 1/4 in their notebooks during individual or a partner work, comes to the board to share their idea.

Jana: (laying the green triangle over the top of the blue triangle)

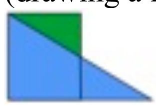
Riya: If you put this one right by here, it would be easy. (laying the blue triangle over the top of the green rectangle) If you cut this part in half, it would be equal to the rectangle. Because-

Teacher: Can people see what she is looking at?

Student: No.

Teacher: Okay, maybe, why don't we take it off the chart, so that we don't have to see all those other shapes for a minute. I will take this away. And just look at the one you're trying to show us. Can you explain to the class what you are looking at?

Riya: (drawing a line that needs to be cut)



Jana: If we cut this (pointing to the extra piece of the blue triangle) in half and would fit in here (pointing to the extra piece of the green rectangle), and it would make them equal.

Teacher: You could do that. We have scissors. Do you understand what they are saying? Can someone repeat what Jana and Riya just said while they take the scissors? Before you do, let's see if people understood you. Wait a second. Can someone explain what they just said about cutting? Sandra?

Sandra: Because if you were to cut the extra piece off, and put it, uhm, over the green part of the rectangle, then you'd be able to see that they're the same area.

Teacher: Is that what you're gonna do, girls?

Riya: (nodding her head)

Jana first puts the green rectangle on the top of the blue triangle, but Riya reverses the order of layers. After repositioning the two shapes, Riya continues her explanation that cutting the blue triangle in half would fit the green rectangle and draws a cutting line on the blue triangle. Jana elaborates on the idea of transforming the cutoff piece from the blue triangle onto the green rectangle. While the teacher gives a scissor to Jana and Riya to cut the extra piece of the blue triangle, she asks other students to repeat what they are doing by cutting. In repeating what Riya and Jana are doing with cutting, Sandra removes the mathematical inaccuracy by replacing "cutting in half" to "cut the extra piece off" and adds mathematical accuracy by replacing "equal" to "same area." After

Sandra's repetition, the teacher checks with Jana and Riya whether it is an accurate description of what they are doing. While Sandra repeats her explanation, Riya cuts the extra piece of blue triangle and rearranges it over the green triangle, which produces a blue rectangle.

Riya:	Uhm, we cut it half, so it would be equal to the rectangle because when we put it under, it wasn't equal shape, but when we cut it, we can make it equal shape. That's why when we put it one areas, they are all equal. That's how we got one-eighth. 
Teacher:	Why don't you ask people first some comments? Say what do you think about that. What do you think about what Riya and Jana did? Does that prove that the green triangle and the green, sorry, the green rectangle and the blue triangle have the same area? Can someone comment on what they did and see what you think about that? You can call on somebody, girls.
Jana:	Uhm… I guess… Levi.
Levi:	Even though the blue triangle doesn't look the same, doesn't look like, it looks like it's bigger than the green, uhm, rectangle. If you can cut it into different parts, it's still-they're still the same. They are still equal to each other.
Teacher:	How about somebody else? Can somebody else try explaining whether you think that what Riya and Jana did shows the green and the blue have the same area? Manley, what would you say?
Manley:	What they did was just they cut the extra part off the triangle and put it on the top of the square and it became the same area.
Teacher:	The rectangle?
Manley:	Yeah.

After cutting the extra piece of the blue triangle and moving it over the green rectangle, Riya continues her explanation that they have the same area, which leads to 1/8. The teacher then asks for comments from other students. Levi comments that the blue triangle looks bigger than the green triangle, but cutting the extra piece proves that they are equal to each other. Manley explains what Jana and Riya did and that the shapes are the same area. At this point, Aiyana asks a question.

Aiyana:	I kind of have a question.
Teacher:	Okay.

Aiyana: Like, like, if you like put the triangle down, instead of like putting the square down, and put the square under the triangle, would there be like a little-be like a little piece like, so be-so like, it'll kind of be like a triangle-

Teacher: -so do you wanna do it the other way around? Is it what you are asking? That's really interesting question. Can everyone understand what Aiyana just asked? Ask it one more time. That's a great question.

Aiyana: Like, could you, uhm… lay a triangle and put the square on top of it and see like if there's a piece like, that can like go to-go to the square that makes the shape of the triangle?

Teacher: So they started with which? What did they put down first? Sandra? What shape did they put down first?

Sandra: Rectangle.

Teacher: The rectangle. And they cut the triangle to fit. What is Aiyana asking? Callie, do you know what Aiyana is asking?

Callie: (silent)

Teacher: No? So, Aiyana, ask it just one more time. She is asking, say what you are asking?

Aiyana: Can you put down the triangle and then put the square on top and then see if a piece can, see if a piece goes with, if you need to cut off a piece of the rectangle, square, and see if it'll go with the triangle.

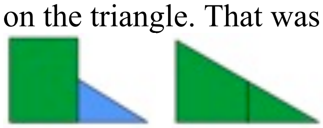
Following Levi's and Manley's comment, Aiyana asks a question about putting the two shapes in the other way around. To make a clear about the idea, the teacher asks Aiyana to repeat her question and checks others' understanding. Elina comes to the board to put the green rectangle on the top of blue triangle and transforms it as Jana and Riya did.

Teacher: Watch and see what Elina is doing and see if what Aiyana asked will work. Aiyana, why did you ask that question? It's a great question. What made you think of it?

Aiyana: I was looking at it, and I had a thought maybe you cut off the square that I can fit into the triangle. Because… hmm.

Teacher: Okay, Elina, tell us what you did.

Elina: I took, I took a piece of this information and a piece of, I took the whole rectangle and then cut it, and then put it over the blue part on the triangle. That was showing.



Teacher: Alanya, is that what you meant?

Aiyana: Yeah.

Teacher: So what you think about that now that you thought?

Aiyana: You can do it with the other way around too and you can do it both ways.

Elina shows that the green rectangle and the blue triangle are the same area by transforming the green rectangle into the green triangle which is congruent to the blue triangle. The teacher asks for comments on Elina or Aiyana. Marlais said that he has another idea of proving. Before diving into Marlais' idea, the teacher asks other students whether they have any other comments. Manley agrees because he would do the same thing. The teacher then asks students to write in their notebooks whether they think that the blue triangle and the green rectangle are the same size, not just the same shape. After individual notebook writing, Marlais comes to the board and put the blue triangles together to make a slightly bigger rectangle. On the top of that, he puts two green rectangles together to make a slightly bigger rectangle. Marlais shows that two blue triangles and two green rectangles take the same area.

Teacher: Okay, so now Marlais, why do you think that prove the same area. Can you explain it?

Marlais: Because they both make a big rectangle and I used two of each, I used two of the blue one and then two of the green ones.

Teacher: Okay. Can people hear what Marlais said? Do you want him to say it one more time? Okay, say it one more time with more like an eight voice.

Marlais: Well, I know the areas are equal because I used two of the blue ones and two of the green ones to make a big square.

Teacher: Rectangle.

Marlais: (nodding his head) rectangle

Teacher: Okay. Now, you can take comments. Ask people if they have comment. Who has a comment for Marlais' method? Does that help to show that the blue and the green have the same area? What do you think?

Marlais: Levi.

Levi: I do think it helps and I think that way is easier than the other ways because you don't have to cut it, you can just put two up next to each other and it's just easier to make.

After Levi provides a comment that Marlais' method is easier than the other two methods, Manley explains why Marlais's method proves that the green rectangle is the same fraction as the blue triangle.

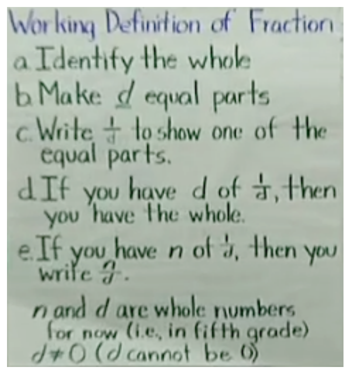
Manley:	Because it fits, because if the green squares, I mean the green rectangles fit on the two triangles and it must have the same area.
Teacher:	That was very clear sentence. Say that again with your eight voice.
Manley:	Because if the, because if the green rectangles fit on the two triangles, they must have the same area.
Teacher:	Natania, what do you think about that?
Natania:	(silent)
Teacher:	Did you hear what Manley says?
Natania:	(silent)
Teacher:	Okay, that time you should have heard it because he used an eight voice. Listen to it this time.
Manley:	Because if the green rectangles fit into the blue triangles, they must have the same area.
Teacher:	(to Marlais) A nice idea, Marlais.
Manley:	Triangles, then it must have the same area.
Teacher:	(to Natania) What do you think?
Natania:	I think he's right.
Teacher:	You think Manley is right? Anyone else wanna ask a question or make comments? Okay. So we've seen three different ways of showing it.
Manley:	(inaudible)
Teacher:	You have-who does?
Manley:	Aiyana.
Teacher:	Okay, Aiyana, you have a question? Yeah. I'm sorry.
Aiyana:	I agree because like earlier when we had put all of the like squares and rectangles on, on there, they are all equal, the same, even though they're different shapes.
Teacher:	Say that again? They are all what?
Aiyana:	They are all equal, the same, even though they're different shapes.
Teacher:	They are all equal the same whole even though different shapes. Okay.

At the end of the class, the teacher reviews the three methods.

- Malik explains the first method suggested by Jana and Riya (laying the blue triangle on the top of the green rectangle, cutting the extra pieces of blue triangle, and making the blue rectangle)

- Nina explains the second method suggested by Alaynna and Elina (laying the green rectangle on the blue triangle, cutting the extra piece of green rectangle, and making the green triangle)
- Jana explains the third method suggested by Marlais (making a slightly bigger rectangle with two blue triangles and a rectangle with two green rectangles and then overlapping them)

The teacher gives students time to change anything they wrote before and moves to suggesting better ideas of naming a fraction and reviews one by one.



Summary

The EML 2009 students initially review one answer (1/8) for the blue triangle and review one answer (1/8) for the green rectangle problem. In the process of comparing the blue triangle and the green rectangle problem, another proposal (1/2) is made for the green rectangle. The proportion of incorrect answers produced by not taking the intended whole (e.g., 1/2 or 1/4) is low. In explaining the blue and green rectangle problem on Day 5 and Day 6, the EML 2009 students initially use the language incorrectly, inaccurately, and inconsistently; grant incorrect geometric names; do not use the comparative language in describing different sizes of the rectangle; and do not build a connection between the numerical representation and the pictorial representation.

To support students' development of mathematical explanation for the blue and green rectangle problem, the teacher briefly clarifies the intended whole while launching the task, provides mathematical supports upon an individual student's request but does not address the issues in a public space; invites the students to the board that allows them to clarify what is being explained and to build a connection between a verbal explanation and a pictorial representation; uses the established knowledge (i.e., definition of fraction) to support students to explain; provides cutouts as resource for the students to prove that each part has equal size and to identify the whole; and replaces the inaccurate language and fills in the missing word so that the students can appropriate the accurate mathematical term.

5.5. The Case of EML 2010

Preview

In the EML 2010, the blue and green rectangle problem is introduced on the first session of Day 3. After a brief review about the two big ideas for naming a fraction (identifying the whole; dividing the whole into equal parts), the teacher posts the blue and green rectangle problem on the board. While launching the problem, the teacher traces the big rectangle with her finger once but does not extensively discuss what the big rectangle refers to at this point. After a 10-minute partner work, the whole-group discussion begins by discussing the green rectangle. For the green rectangle, Thailee and Samara propose 1/4 and then Madeline and Chanika propose 1/8. The class examines the difference between the whole that Thailee and Samara used and the whole that Madeline and Chanika used. After Madeline and Chanika's presentation, Bernard challenges the method of dissecting the left side of the big rectangle into four triangles and the right side of the big rectangle into four rectangles. For the blue triangle, Michael, Qayshawn, and Terrence propose 1/8. After proving that the green rectangle is 1/8 and the blue triangle is 1/8, the teacher further challenges why they are both 1/8 despite the different shapes. Coretta and Ella transform the blue triangle into the blue rectangle that is congruent to the green rectangle and Jaclyn transforms the green rectangle into the green triangle that is congruent to the blue triangle. An extensive detailed analysis of the 60-minute of instructional interactions managed by the teacher, Ms. Ball, for teaching the blue and green rectangle problem in the EML 2010 is provided below.

Extensive Detailed Analysis

The blue and green rectangle problem is introduced on the first session of Day 3 in the EML 2010. After doing the warm-up problem of Day 3, the teacher makes a transition into the blue and green rectangle problem by reviewing the two big ideas about naming a fraction which were developed from the previous day's discussion of the brown rectangle problem: (1) identify the whole; and (2) to identify the fraction, divide the whole into equal parts. The teacher reminds students that the idea of "equal" parts is very important for naming a fraction. She then sets up that the blue and green rectangle problem provides an opportunity to figure out equal parts and an opportunity to learn more ideas about naming a fraction correctly. After putting the poster on the board, the teacher asks students to read each question.

What fraction of the big rectangle is shaded green?
What fraction of the big rectangle is shaded blue?

After Madeline reads the first problem and Jason reads the second problem, the teacher reviews the two big ideas about naming a fraction written on the poster once again. While launching the problem, like in the EML 2009, the teacher traces the whole with her finger once, but does not extensively clarify what the big rectangle refers to at this point. The teacher assigns a partner to each student so that they could work on the problem together. Table 5.5 shows the answers that students produced for the blue and green rectangle problem in their notebooks. More than half of students (15 out of 28 students) clearly recorded the correct answer of 1/8 in their notebooks and nine students recorded the incorrect answers (six students wrote 1/4 and three students wrote 1/2) by taking a part of the intended whole. All of the students produce the same answer for the blue triangle and for the green rectangle.

Table 5.5. Answers that the EML 2010 students wrote in their notebooks for the blue and green rectangle problem

Answers		The number of students
Same answer for each question	1/8	15
	1/4	6 (Coretta, Jaclyn, Karina, Samara, Shar, Thailee)
	1/2	3 (Anthony, Jason, Macaulay)
	1/6	1 (Zahara)
	1/5	3 (Devante, Mustafa, Ahmed)
Different answer for each question		0
Unrecognizable		0
No records or absent		0

During the partner work, the teacher circulates the classroom to check whether partners are working together, whether partners agrees on their answer, and whether the students use the working definition of fraction written on the board. The teacher does not voluntarily provide substantial mathematical supports to students but provides mathematical supports upon the requests by students. For example, after a short discussion with his partner Karl, Bernard has a conjecture that the blue triangle and the green rectangle have the same amount. To confirm the idea, Bernard asks the teacher whether the blue triangle and the green rectangle would be equal. Upon Bernard's request, rather than providing a direct answer to his question, the teacher supports Bernard in devising a method to prove his conjecture and checks for Karl's understanding of Bernard's idea.

After a 10-minute partner work, the teacher reconvenes the class to begin a whole-group discussion. Thailee and Samara get a turn to explain their answer for naming a fraction that is shaded in green on the board. Figure 5.7 and Figure 5.8 illustrate their written explanations in their notebooks. It is interesting to see that both Thailee and Samara changed the problem statement from "the big rectangle" to "the small rectangle" in their notebooks. Both drew a longer line in the middle and then dissected each side with the same shapes (the left side of the big rectangle is dissected into triangles which are congruent with the blue triangle and the right side of the big rectangle is dissected into the rectangles which are congruent with the green rectangle).

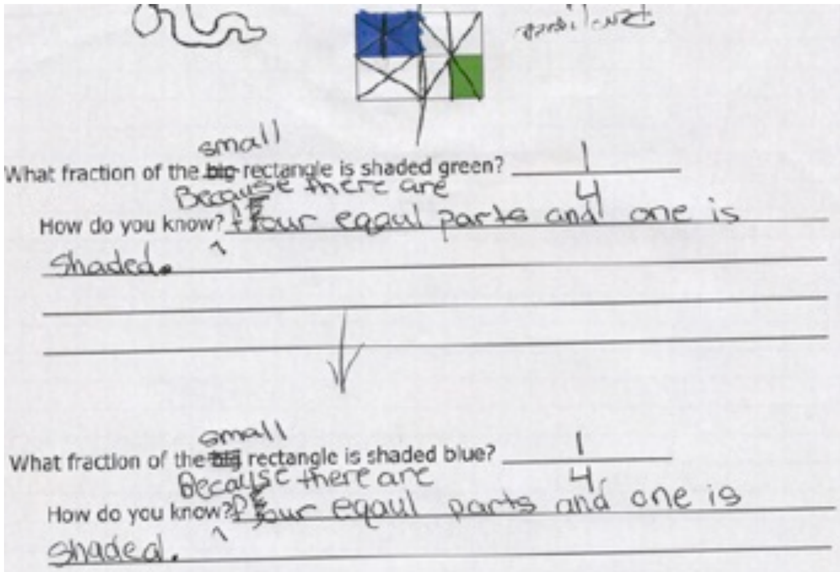


Figure 5.7. Thailee's notebook writing for the blue and green rectangle problem

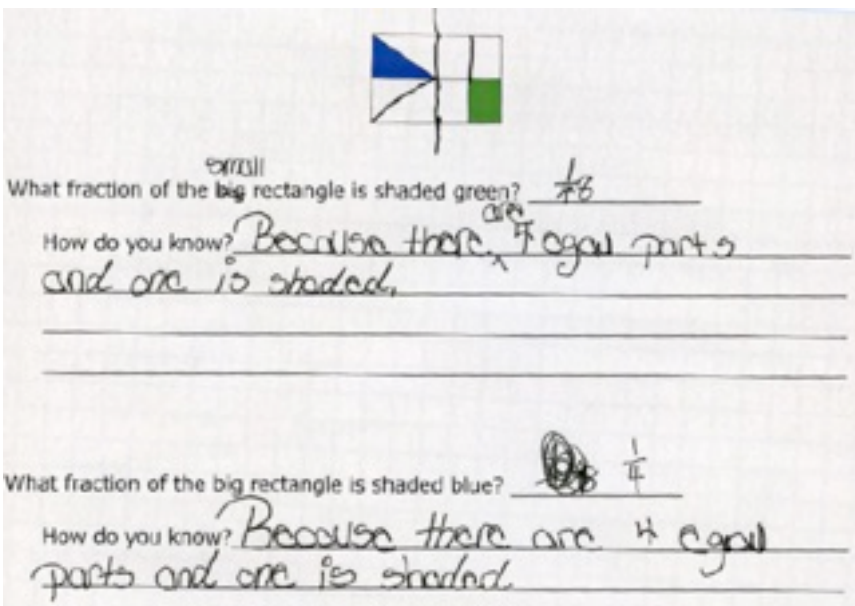
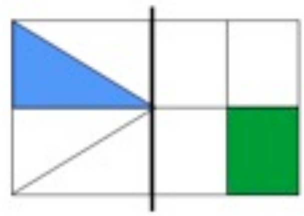


Figure 5.8. Samara's notebook writing for the blue and green rectangle problem

Thailee:	Uhm, we decided that we would split the big rectangle and make it two small rectangles and put a line right here to make it four equal parts and make another-
Teacher:	-You can go ahead and do it if you want. You can write on that.
Thailee:	You wanna do the writing?
Samara:	Sure.
Thailee:	Uhm, and we made four squares (four little rectangles on the right side of the big rectangle) and then we made two small rectangles (pointing the line in the middle which divides the left-side

rectangle and the right-side rectangle), made four rectangles (four little rectangles on the right side of the big rectangle) and four triangles. And both of them ended up being one-fourth.



Teacher: So, what was your answer to the green? How much of the big rectangle is shaded green? What did you decide?
Thailee: One-fourth.
Teacher: One-fourth? Can you write down one-fourth next to your drawing?
Samara: (writing 1/4 on the board)
Teacher: Okay, the next thing we're gonna do is nobody's gonna decide if they agree or disagree yet. I just want you ask question to make sure you understand what they said. Who has a question for them about what they said? Don't decide if you agree or disagree with them. Do you have any questions about what they said?

Thailee first divides the big rectangle into two small rectangles (the left side of the big rectangle and the right side of the big rectangle). After that, she adds a line in the upper right side of the big rectangle to make four equal little rectangles. Prompted by the teacher's comment, Thailee draws three lines on the drawing: a vertical line in the middle to divide the big rectangle into two small rectangles, another vertical line in the upper right side of the big rectangle to divide the small rectangle on the right side into equal shapes like the green rectangle; the diagonal in the lower left side of the big rectangle to divide the small rectangle on the left side into equal shapes like the blue triangle. In Thailee's initial explanation, three things are noticeable. First, the reference of the "big" rectangle that Thailee used is consistent with the intended whole in the problem statement. Thailee recognizes what the big rectangle refers to in the problem statement, but dissects the big rectangle into "two small rectangles" and then takes the small rectangle as a whole instead of taking the big rectangle as a whole. Second, Thailee assigns different names for different sizes of the rectangles: the big rectangle, two small rectangles, and four rectangles. Third, Thailee makes explicit about making equal parts in her initial explanation. After drawing lines, Thailee continues her explanation that she makes the big rectangle into two small rectangles and then makes one small rectangle into four rectangles and the other small rectangle into four triangles. She then concludes that both shapes are 1/4. The teacher re-checks with Thailee about her answer and Thailee confirms that the green rectangle is 1/4 of the big rectangle. After hearing Thailee's explanation, the teacher checks whether other students have a question for Thailee and specifically asks them to neither agree nor disagree with Thailee's idea. As no one asks a question, the teacher proceeds to ask Thailee a question. Making a reference to the ideas on the steps for identifying a fraction, the teacher asks Thailee and Samara to identify the whole.

Thailee:	Uhm, we made two-uhm, instead of one big rectangle (tracing the big rectangle) where it would be an even, we made it two small rectangles (pointing her finger to the middle of rectangle to show the split)
Teacher:	Can you speak up a little bit Thailee?
Thailee:	We made it two small rectangles.
Teacher:	So draw your finger around the part that you're calling the whole. Can you do that?
Thailee:	(tracing the right side of the rectangle and tracing the left side of the rectangle)
Teacher:	Okay. Did everybody see what Thailee just did?
Students:	Uh-huh.
Teacher:	Do it one more time, Thailee.
Thailee:	(tracing the right side of the rectangle and tracing the left side of the rectangle)
Teacher:	So who can answer-what did Thailee and Samara decide was the whole for answering the first question? Who can describe what they did? Coretta?
Coretta:	They split the whole, the right side of the rectangle into four little rectangles and they called it the whole because they are all equal.
Teacher:	Okay, you gave a very good complete explanation. I'm gonna repeat it. Okay? Coretta said that what the girls did is they divided the right side of the big rectangle and made it into four equal pieces and called that the whole. Is that what you said?
Coretta:	(nodding her head)
Teacher:	Is that what you did girls?

Thailee and Samara: (nodding their heads)

Thailee explains that she takes two small rectangles instead of one big rectangle. Even though her naming of the big rectangle is consistent with the big rectangle in the problem statement, she explains that she takes two small rectangles as wholes instead of one big rectangle as a whole. To clarify the whole that Thailee takes, the teacher asks her to trace the whole and Thailee traces the edge of the left side of the rectangle and the edge of the right side of the rectangle. The then teacher asks Coretta to describe what Thailee and Samara did, repeats the explanation once again, and then checks back with Coretta, Thailee, and Samara.

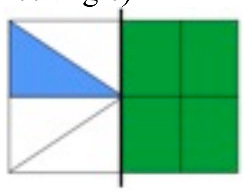
Teacher:	Okay, after you did that, what did you do? So then you identified the whole and then to identify the fraction, you divided the whole into equal parts. Can you show us the equal parts you made on that whole?
Thailee:	Uhm this one (pointing to the four small rectangles on the right side of big rectangle), this one (pointing to the four small triangles on the left side of the big rectangle)
Teacher:	And how—wait a minute, I thought you said just the right hand side was the whole. Which side is the whole?
Samara:	No.
Thailee:	Both of them are wholes.
Teacher:	You have two wholes?
Thailee:	Yeah.
Teacher:	So fix-focus on the first one first. Cause you have the first one when you talked about. So, for the green question, you looked at the one on the right?
Thailee:	Uh-huh.
Teacher:	So count how many are in that whole?
Thailee:	(pointing four little rectangles on the right-hand side of the rectangle) One, two, three, four.

After clarifying the whole that Thailee and Samara used, the teacher moves to the second idea of naming a fraction: dividing the whole into equal parts. Thailee first points to the four small rectangles on the right side of the big rectangle and then points to the four small triangles on the left side of the rectangle. The teacher asks which one is the whole and Thailee responds that both of them are wholes. Because Thailee uses the right side of the big rectangle as a whole for the green rectangle and uses the left side of the big rectangle as a whole for the blue triangle, the teacher fixes the right side of the rectangle and then asks her to count how many are in Thailee's whole. The next step of explaining is to show how many equal parts in Thailee's whole. The teacher gives Thailee sticky green rectangle cutouts so that she can cover her whole with four green rectangles.

Teacher:	Okay. So, I'm gonna ask you to show us that there're four equal parts and I'm gonna give you some pieces that will let you show it.

Here's some more pieces that are the same as the original one (handing the green rectangle sticky cutouts). Can you show us that those are all the same and fill up your rectangle? Here, Samara, you can hold. So I gave them something that will help them prove to us that all—those are four equal parts.

Thailee: (Putting the sticky green rectangles on the right side of the big rectangle)



Teacher: Okay, so what did you show us now, Samara?

Samara: Huh?

Teacher: What does that show?

Samara: Four equal parts.

Teacher: Do people agree that that show four equal parts?

Students: Yes.

Teacher: Out of what whole? Can someone say what whole Samara and Thailee were working on? Ahmed?

Ahmed: whole

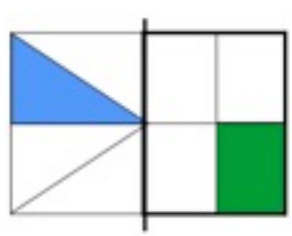
Teacher: Where's the whole? Can you come up and show us where the whole is that they are looking at? Put your finger around what they are calling the whole.

Ahmed: (coming up to the board and tracing the right side of the big rectangle with his finger on the board)

Teacher: Is that right, girls? Is that what you are calling the whole?

Thailee and Samara: (nodding their heads) Uh-huh.

By covering the whole with the same shapes, four equal parts are proven and the whole is rechecked. After that, the teacher asks what they call the green rectangle. Eric explains that they call the green rectangle 1/4 because "there was four equal parts and one was shaded." The teacher draws Thailee's whole with the marker so that they easily identify what Thailee calls the whole.



The teacher then asks another pair of students who calls a different whole than that identified by Thailee by reminding them that identifying the whole is a very important idea for naming a fraction. The teacher calls Madeline and Chanika to come up to the board and hints that the first step is to identify the whole.

Madeline: Uhm, the whole that we did was eight, because we split it all in half

Teacher: Can you talk to the class please?

Madeline: Uhm, the whole we did was eight, we split it all in half and we got eight.

Teacher: So could you put your finger around what you identified as the whole?

Madeline: (tracing the whole rectangle)

Teacher: Who saw what Madeline just did? Great! Can someone explain the difference between what Madeline did and what Thailee did? They did different things when they identified the whole. What was different, Michael?

Michael: Uhm, that's Thailee and Samara did half of it same whole and Madeline, uhm-

Teacher: Chanika-

Michael: Did the whole thing.

Teacher: Okay, so Madeline and Chanika, is this correct girls? Are you using this rectangle (tracing the big rectangle) to be their whole, whereas what Thailee and Samara did- you used this rectangle to be the whole. Okay, so keep going. Now, what did you do? So you have to divide that whole into equal parts. Can you show us how you did it?

Madeline and Chanika: (after drawing the vertical line on the upper right side of the rectangle, handing the pen to Chanika and whispered "half this way" by tracing the diagonal with her finger to Chanika)

Madeline: And then we counted them and we got eight.

Teacher: Can you count them for us?

Madeline: One, two, three, four, five, six, seven, eight.

Teacher: Who can describe what they just did? Elias?

Elias: Uhm, they put the four squares in half—uhm like how the blue and red-no blue and green

Teacher: Talk a little louder, Elias.

Elias: They split the four squares in half to make eight equal parts.

Teacher: Okay. Does everyone agree with Elias that girls cut the four squares that are inside the big rectangle, each in half and then counted eight?

Students: Yes.

Madeline initially explains that “the whole that we did was eight” instead of “the whole that we did was composed of eight parts.” In addition, she elaborates that she splits it all in half but does not specify what “it” refers to. When Madeline provides information about the number of pieces and how she gets them, the teacher asks Madeline to trace her whole. Afterwards, the teacher asks about the difference between Madeline’s whole and Thailee’s whole, and Michael provides an explanation. Madeline and Chanika add lines, which are quite similar to Thailee and Samara’s. The teacher asks Elias to describe what Madeline and Chanika did and Elias fills up the missing information in Madeline and Chanika’s explanation in an ordered way (providing information about splitting four squares in half first) and adds that the eight parts are equal.

Teacher:	Now, what’s the question to ask them? It’s a pretty good question you need to ask them. And I wanna see if you know what question to ask. We’re not agreeing or disagreeing. I want you to learn to ask a question. There’s something there that you should ask them about. What is it? Bernard, what should you ask them?
Bernard:	Why didn’t you cut all—all the squares into rectangles or cut all the squares into triangles?
Teacher:	Can you say that a little louder? That’s a great question, but everyone couldn’t hear it.
Bernard:	Why didn’t you cut the rectangle-the-all the squares into rectangles or cut all the squares into triangles?
Teacher:	And why did you ask that question?
Bernard:	Because you could-you can either cut them into triangles and divide them like that or cut them into rectangles and then use them like that.
Teacher:	And why is that -why are you wondering about that, Bernard? That’s a good question. Why are you wondering? Cause I know you and Karl talked about that too.
Bernard:	Because if you divide them into rectangles, you have-the triangles wouldn’t fit into the rectangle. If you divide them into triangles, the rectangle wouldn’t fit into the triangle.
Teacher:	So, Bernard is pointing out the problem. We wanna make sure all these parts are equal… and I’m adding a little bit to what Bernard said. And if some of them are triangles and some of them are rectangles, can we be sure that those are equal size parts?
Students:	No.
Teacher:	Is that what you are pointing out? So, that’s kind of an interesting question. Do you girls think that you can answer that? Do you think that you believe that every one of those eights are equal areas?

Madeline: Well the only reason we didn't do that is because I thought if we didn't-if we did it straight (pointing to the left-hand side of the rectangle), it wouldn't be equal as the….

In a similar way, the teacher asks students to neither agree nor disagree with the answer that Madeline and Chanika suggested, but calls for question that students should ask. Bernard challenges the method that Madeline and Chanika used for dividing the whole into equal parts. Figure 5.9 shows what Bernard wrote in his notebook during a partner work.

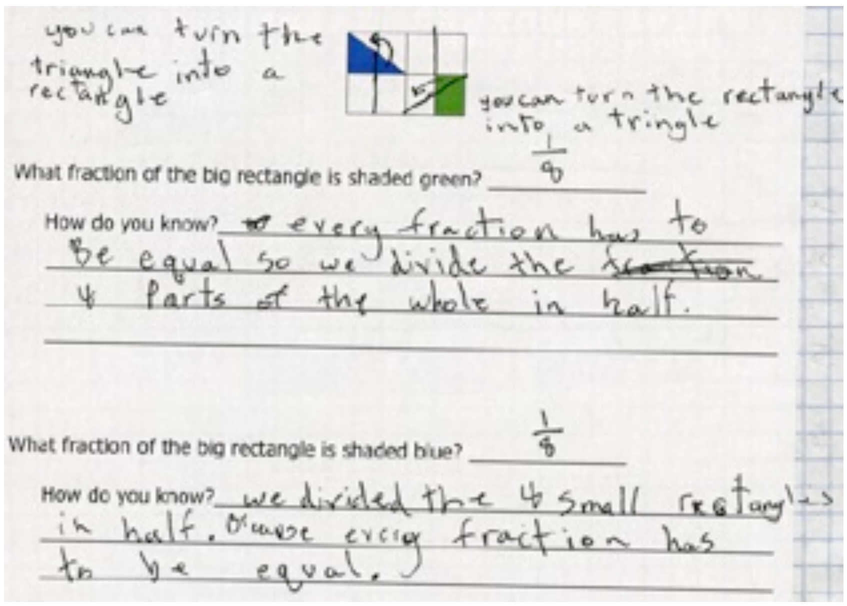


Figure 5.9. Bernard's notebook writing for the blue and green rectangle problem

In the previous exchange, it is not problematic to see that Thailee divides her whole into equal parts because all parts of the whole have the same shape as green rectangle. However, in Madeline and Chanika's drawing, even though the way of cutting is quite similar to Thailee and Samara's, it might be questionable in seeing whether Madeline's whole is divided into equal parts because four of them are triangles and four of them are rectangles. Thus, in order to accept the idea that Madeline's whole is divided into equal parts, a further explanation is needed. The teacher then asks Madeline and Chanika to use the sticky green rectangle cutouts to fill the whole. In the meantime, the teacher asks how many greens it would take to fill Madeline's whole and how many greens it would take to fill Thailee's whole and asks for a reason why it takes different number of greens to fill Madeline's whole compared to Thailee's whole. To show that each of the eight greens has the same area, Madeline overlaps two sticky cutouts of green triangles as Ella suggested.

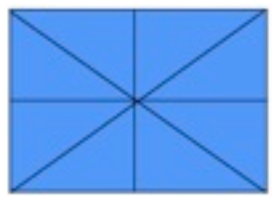
The teacher asks students to talk with a partner why they get two different answers for the green rectangle: Thailee and Samara said 1/4, whereas Madeline and Chanika said 1/8. After a brief exchange with a partner, the class examines together why Thailee and Samara call the green rectangle as 1/4 and Bernard explains why Madeline and Chanika call the green rectangle as 1/8. At the end of these exchanges, Thailee adds comments about their whole.

> Thailee: Uhm, yeah, we have two wholes.
> Teacher: You have two wholes. And we're getting back to your other whole now. And we'll get back to it. Cause now we're gonna talk about the blue.

Hearing that Thailee uses the right side of the rectangle as a whole, the teacher sets forth a counterargument that they have two wholes instead of one whole. Thailee and Samara use two wholes, with one whole applying to the green rectangle and the other whole applying to the blue rectangle. The teacher points out that they have been discussing a whole, which is used to figure out what fraction of the green rectangle is shaded. After surveying how many students get the same whole as Thailee and Samara and how many students have the same whole as Madeline and Chanika, the teacher makes a transition to the blue triangle.

The teacher gives Michael, Qayshawn, and Terrence a turn to explain what fraction is shaded blue and frames that the first step is to tell what they call the whole. As Michael identifies their whole by tracing the big rectangle, the teacher checks whether they divide the whole into equal parts. Michael draws diagonal lines to make equal parts and then counts eight equal parts.

Different from the other two groups, who added lines but did not touch the other color, Michael draws a diagonal line over the green rectangle, but makes all into the equal shapes. Michael then tries to cover their whole with the sticky blue triangles. Meanwhile, the teacher reviews what they call the whole and whether their whole is similar to Thailee and Samara or Chanika and Madeline. Javonte answers that Michael's whole is like Chanika and Madeline's whole and described it as a "big whole." During these exchanges, Michael just covers the whole into eight blue triangles.



Qayshawn explains that they put eight sticky blue cutouts in the whole. When the teacher asks about the answer, Qayshawn answers 1/8 but could not go further in explaining how he gets 1/8 in a full sentence. The teacher makes a reference to the poster and checks what they identify as a whole and asks them to show equal parts. The teacher then asks Terrence to prove whether those blue triangles are equal.

Teacher:	Okay. Now Terrence, I'm gonna ask you a question. Can you help us to see how you guys decided those were equal? And not just eight things? How do you know those eight parts are equal?
Terrence:	Because the triangles, they're like all the same size.
Teacher:	Say that again?
Terrence:	The triangles are all the same size.
Teacher:	Can you show us somehow prove to us those are all equal size?
Terrence:	Well, the line in the middle is like, not, like, it's not closer to this (drawing a line with the finger to the left side) and it's not closer to that (drawing a line with his finger to the right side), so you can tell it's…
Teacher:	Say that to the class. That was good, but say that to them.
Terrence:	The line in the middle is, uhm, not to close to this, and it's not too close to that. So that's how you know it's equal.
Teacher:	Do people agree? Are those equal areas? Each of those blue triangles?

Students: Yes.

It is quite clear that all of the blue triangles are the same size. For the green rectangles, Madeline overlaps two green rectangles to prove that they are equal as Ella suggested. Terrence proves that the diagonal line dissects the rectangle in half, which connects the opposite vertices. By covering the whole with eight blue triangles of the same size, Terrence proves that the whole is eight-eighths and one of them is one-eighth. While the teacher goes over again what Michael, Qayshawn, and Torrece did, the teacher helps Ella complete an explanation. The teacher points out that Thailee and Samara called something different as a whole than the intended whole in the original problem and seeks for an agreement on the whole in the original problem. After proving that each shape is 1/8 of the whole, the teacher asks how both the blue triangle and the green rectangle are 1/8 while pointing out that they look different (not the same shape). The teacher then gives four minutes for students to talk with a partner and writes the new question on the board: "How could the blue triangle and the green rectangle each be one-eighth of the big rectangle? Even though they are not the same shape. Can you prove it?"

Ella: The reason we thought that they could be the same shape is because, uhm, if you cut it like the rectangle (drawing the vertical line in the upper-left side of the big rectangle with her finger), this part, uhm, the part that's on this side would be able to flip up here-

Teacher: Why don't you take a blue one and do it? Do you need scissors? Can someone give them a pair of scissors? Here, girls, here's a scissor.

Students: (talking each other)

Teacher: Just stop a second. Do you understand what they're going to try to do? They're gonna try to what? What are they going to try to make out of the blue triangle? Can someone describe what they're going to try to make? Yes, Thailee?

Thailee: Squa- a rectangle.

Teacher: They will look like what?

Thailee: (silent)

Teacher: What will it look like when we're done if they're right?

Thailee: A rectangle.

Teacher: Which one though? It's gonna look like the-

Thailee: The… triangle.

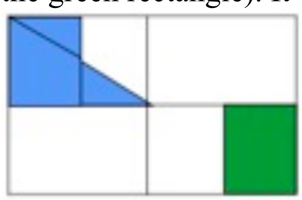
Teacher: The blue rec, triangle will look like the-

Thailee: -rec, the green rectangle.

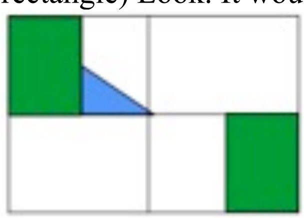
Teacher: The green rectangle. Exactly. So, they're gonna try to prove they can build the blue-the green rectangle out of the blue triangle by cutting it.

The teacher challenges students that the blue triangle and the green rectangle are different shapes, but Ella explains a way to make them into the same shape by cutting a piece of blue triangle and rotating the cut-off piece to make the same shape like a green rectangle. After Ella sketches a way of proving to make them into the same shape, the teacher checks whether other students follow up with Ella's idea. Even before Ella cuts the piece of the blue triangle and transforms it, Thailee provides a picture that the blue triangle will look like the green rectangle.

Coretta: What we're trying to show is, if you cut this (pointing to the blue triangle) in half, it would be almost like the rectangle (pointing to the green rectangle). It would be almost like the rectangle.

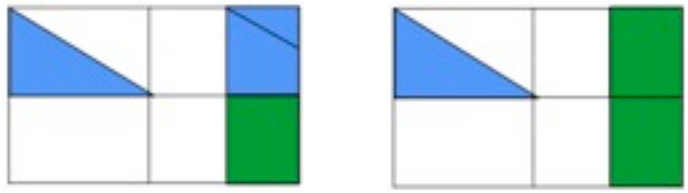


Ella: (bring another green rectangle and put it over the transformed blue rectangle) Look. It would fit…



Coretta: The rectangle.
Ella: Almost perfectly.

Coretta describes that she cuts the blue triangle in half, not in terms of cutting the area of the blue triangle in half, but the cutting line is the middle point of one of the bases, and that it is almost like the rectangle. To show that the transformed blue triangle is the same shape of the green rectangle, Ella overlaps the green rectangle over the transformed blue rectangle. The original blue triangle is colored on the paper, so they move the blue triangle to the next space and explains once more.



Coretta:	We're putting a rec, we're putting a piece of the triangle together from where it was slanted, and half of the square. Then we cut a piece off and if you put the rectangle on top, and it looks exactly, like, it looks exactly like the bottom rectangle (Ella put the green rectangle over the blue rectangle). And you can make it even by that way or you can do it different way.
Teacher:	So they're trying to show that, if you cut the blue triangle up, you can make a shape that looks just like the green rectangle and prove that they're the same size. So, what do you think about that? Do you think that proves that you can, those are the same size, since you can cut them and make them and turn them into the same shape? Talk to your partner and see if you think that that's pretty convincing or not.

Despite the inaccurate terms ("the square" instead of "the rectangle" or "even" instead of "equal"), Coretta clearly lays out the ideas that she assembles two pieces of the blue triangle to make a congruent shape to the green triangle. After Coretta's explanation, the teacher makes space for individual students to reflect on whether they are convinced by Coretta's method.

Teacher:	(to the whole class) Okay, so look up again. Can someone explain what Ella and Coretta tried to show by cutting? What were they trying to show us? When I asked how could the blue triangle and the green rectangle each be one-eighth of the big rectangle, even though they're not the same shape? What did they try to show us? How come I see so few hands? What did they try to show? They came up and they did something to try to prove something. What did they try to prove to us? Michael?
Michael:	That they could make, uhm, the green rectangle with the triangle by cutting it up.
Teacher:	Okay. That's a very nice statement. They tried to prove that they could make the green rectangle out of the blue triangle by cutting it up. Does that prove that they are the same size? Why is that, Javonte? Why does that prove they're the same size?

After Michael's restatement, the teacher continuously checks whether the method proves that the blue triangle and the green rectangle are the same size. With the teacher's continuous probing, Jaclyn shares another way of proving by cutting up the green rectangle to make a congruent shape as the blue triangle and Thailee shares her idea about putting two shapes together to make a rectangle. Jaclyn shows how she could make a green triangle out of the green rectangle on the board.

At this point, Thailee asks whether she could share her method, but the time constraint does not allow her to share another way of proving. It is almost time for the break, so the teacher wraps up the lesson. After the break, the teacher moves onto reviewing the working definition of fraction.

1. Identify the whole
2. To identify the fraction, divide the whole into equal parts
3. When the whole is divided into d equal parts, we call one of the equal parts 1/d.
4. When we have all the equal parts, it is the whole, and we write d/d.

Summary

The EML 2010 students propose two answers (1/4 and 1/8) for the green rectangle and propose one answer (1/8) for the blue triangle. The proportion of the students who recorded the incorrect answers by not taking the intended whole (e.g., 1/2 and 1/4) is high. The presented method, which does not dissect the whole into the same shapes, is challenged by the students. In explaining the blue and green rectangle problem on Day 3, the ELM 2010 students initially use the language incorrectly, inaccurately, and inconsistently; grant incorrect geometric names; do not use the comparative language in describing different sizes of the rectangle; do not build a connection between the numerical representation and the pictorial representation.

To support students' development of mathematical explanation for the blue and green rectangle problem, the teacher provides mathematical supports upon an individual student's request but does not address the issues in a public space; invites the students to the board that allows them to clarify what is being explained and to build a connection between a verbal explanation and a pictorial representation; uses the established knowledge (i.e., definition of fraction) to support students to explain; uses the cutout to support the students in proving that each part has equal size and in identifying the whole; and replaces inaccurate language and fills in the missing words so that the students can appropriate the accurate mathematical term.

5.6. The Case of EML 2013

Preview

In the EML 2013, the blue and green rectangle problem is introduced on the second session of Day 3. The teacher introduces the blue and green rectangle problem, but does not clarify what the big rectangle refers to. While introducing the problem, the teacher reminds students to use the steps for naming a fraction written on the board. After an eight-minute individual or partner work, the whole-group discussion begins by discussing the green rectangle. For the green rectangle, Tenisha explains how she gets 1/8 by drawing the vertical lines on the upper right corner of the big rectangle and on the lower left corner of the big rectangle. After Tenisha's explanation, Deshawn requests for repeating and Renee challenges the method of dissecting that all parts are not divided into equal shapes. Faced with the challenge raised by Renee, the teacher gives a turn to Connor to cover the whole with the green rectangles and prove that one green rectangle is called 1/8. The second session of Day 3 is almost over at this point, so the teacher asks students to review their answers for the blue triangle.

On the next day, the teacher revisits the blue and green rectangle problem. After reviewing the work that Connor did on the previous day, the teacher asks Liberty to cover the whole with the blue triangles and to prove that one blue triangle is called 1/8. After giving a turn to Ty to re-explain why the green rectangle is 1/8 and to Kalvin why the blue triangle is 1/8, the teacher further challenges why they have the same fractional name despite the different shapes. The extensive detailed analysis of 21-mintues instructional interactions on Day 3 and 27-minute instructional interaction on Day 4 managed by the teacher, Ms. Ball, for teaching the blue and green rectangle problem in the EML 2013 is provided below.

Extensive Detailed Analysis

The blue and green rectangle problem is introduced on the second session of Day 3 in the EML 2013. The teacher reminds students to use the steps for naming a fraction that are written on the board, but does not provide any clarification about the problem statements written on the poster. In the poster, the triangle on the upper left corner of the big rectangle is shaded in blue and the rectangle on the lower right corner of the big rectangle is shaded in green.

What fraction of the big rectangle is shaded green?
What fraction of the big rectangle is shade blue?

The students work on the problem either individually or with a partner. When the students begin to work on the problem, the teacher draws the students' attention again to address that the blue and green problem is tricky so other fifth graders have been fooled by the problem and reminds them use the steps for naming a fraction written on the board. During the eight-minute independent work, either by individually or with a partner, the teacher supports students in using the steps for naming a fraction written on the poster and to make lines so that the whole is divided into equal sizes.

In their notebooks, 17 out of 29 students have a clear record of 1/8 for the green rectangle and for the blue triangle. Two students wrote 1/4 and one student wrote 2/8. Unlike the previous cohorts, five students recorded different answers for the blue triangle and for the green rectangle. Among them, Madeline, Anaya, and Renee produce answers that are interesting. All of three students have shown that they understand the key ideas for naming a fraction. They produce the correct answer (1/8) for one color but produce the incorrect answer (2/8) for the other color. Depending on the partitioning method, they incorrectly name one shaded part (if they divide the rectangle into triangles that are congruent to the blue triangle, they do not name correctly the green rectangle; if they divide the rectangle into rectangles that are congruent to the green rectangle, they do not name correctly the blue triangle)

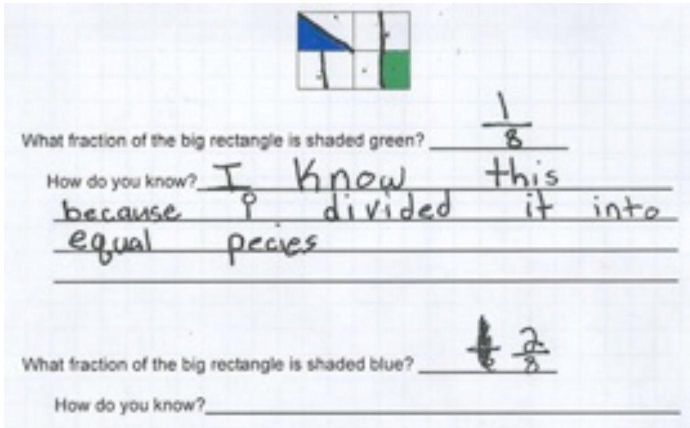


Figure 5.10. Madeline's notebook writing for the blue and green rectangle problem

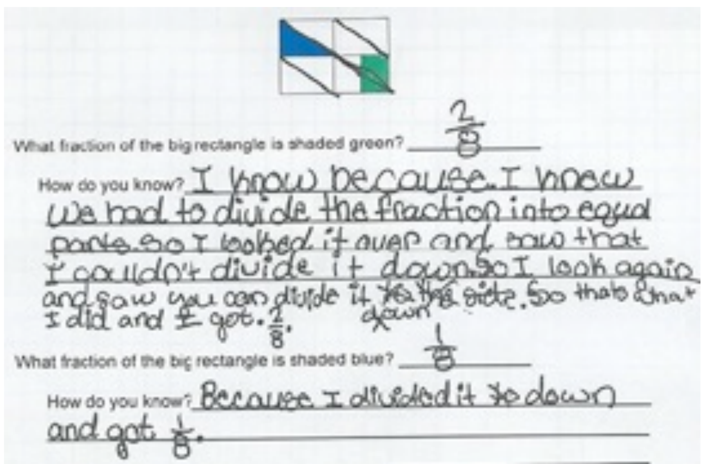


Figure 5.11. Anaya's notebook for the blue and green rectangle problem

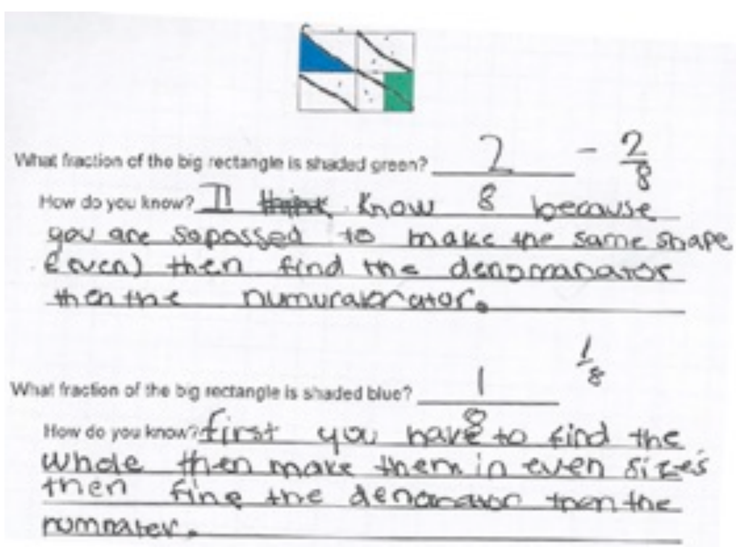


Figure 5.12. Renee's notebook writing for the blue and green rectangle problem

Table 5.6. Answers that the EML 2013 students wrote in their notebooks for the blue and green rectangle problem

Answers		The number of students
Same answer for each question	1/8	17 (Ahmed, Aziz, Tina, Deshawn, Demonte, David, Aryanna, Liberty, Ty, Kallie, Kalvin, Bria, Mark, Elysa, Tenisha, Isabella)
	1/4	2 (April, D'lon)
	2/8	1 (Tashawnah)
Different answer for each question	1/8 for green and 2/8 for blue	1 (Madeline)
	2/8 for green and 1/8 for blue	2 (Anaya, Renee)
	1/8 for green and 1/2 for the blue	1 (Louis)
	1/6 for green and 8/8 for blue	1 (Connor)
Only one answer	1/8 for green	2 (Jarvaise, Otis)
Unrecognizable		2 (Nicholas, Elijah)
No records or absent		0

After eight-minute individual or partner work, the teacher re-convenes the class to open a whole group discussion. Reassuring that the problem is very tricky, the teacher introduces the steps for naming a fraction that are written on the board, sticky blue triangle cutouts and sticky green rectangle cutouts. Tenisha, who wrote 1/8 for the blue triangle and 1/8 for the green rectangle in her notebook, comes to the board to explain her answer. In the notebook, Tenisha recorded:

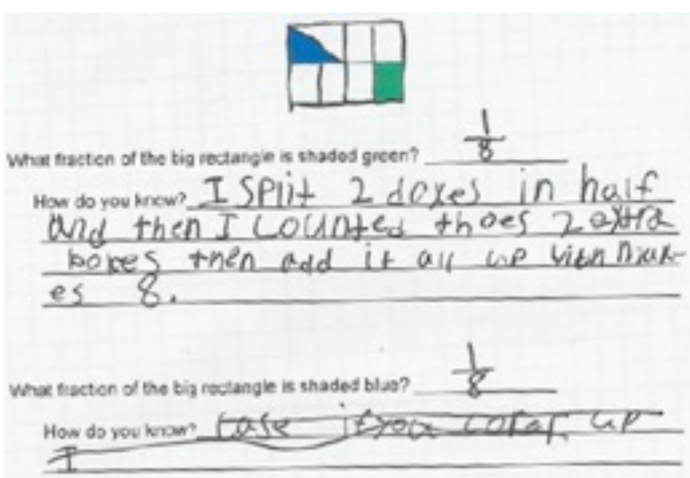


Figure 5.13. Tenisha's notebook writing for the blue and green rectangle problem in her notebook

Tenisha:	First, I cut that one in the middle (showing the gesture to draw a line on the upper right part of the rectangle)-
Teacher:	Wait, go back to the chart (pointing to the “steps for naming a fraction” chart) for a minute, Tenisha. Start with the chart next to you, Tenisha. Figure out-
Tenisha:	This one? (pointing to the “steps for naming a fraction” poster) Okay. Figure out what the whole is. Make sure-
Teacher:	So, where’s the whole? Can you show us?
Tenisha:	The whole?
Teacher:	The whole rectangle (showing the gesture to make a whole rectangle). Can you show us? With your finger?
Tenisha:	(pointing to the upper right part of the rectangle)
Teacher:	The whole shape (gesturing to make a whole rectangle), Tenisha.
Tenisha:	Oh. (tracing the whole rectangle with her finger)

Tenisha begins her explanation by gesturing how she cuts the upper right side of the big rectangle. Before Tenisha finishes describing about her method, the teacher asks Tenisha to use the steps for naming a fraction. Tenisha reads the first step written on the steps for naming a fraction and traces the whole with her finger. Tenisha first points to the upper right side of the big rectangle but then traces the big rectangle that she uses to get her answer. After that, the teacher guides Tenisha in following the next step of naming a fraction.

Teacher:	Good. What is the step two saying?
Tenisha:	Make sure that the whole is divided into equal parts. So, I can make that one half (gesturing to draw a vertical line in the middle of the upper right side of rectangle) and that one half (gesturing to draw a vertical line in the middle of the lower left side of rectangle)

Tenisha gestures by drawing lines for the pieces that are not divided yet (the upper right side of the rectangle and the lower left side of the rectangle). By making a reference to the steps for naming a fraction, Tenisha conveys the idea of making the whole into equal parts even though she does not explicitly address it in her explanation. Her use of “one-half” would be translated to making it equal parts rather than naming 1/2 as a fraction for each divided piece. Tenisha then draws lines with a marker.

Tenisha reads the third step for naming a fraction—count how many equal parts are there—and answers eight. She then count eight parts by pointing to each of them one by one and calls one of the equal parts 1/8.

Tenisha: So I divided those two (points to the upper right side of the big rectangle) in half and those two in half (points to the lower left side of the big rectangle) and I counted all up which I got one-eighth.
Teacher: Okay, ask for some comments. Say "Do you have comments?"

After completing the paritioning, Tami re-explains how she gets 1/8 for the green rectangle. Tenisha chooses the language of "dividing those two in half" which conveys the idea of making them into equal parts. Instead of further probing or asking to repeat, the teacher elicits comments from other studetns. Deshawn, who wrote 1/8 for the green rectangle problem but did not write an explanation in his notebook, asks for explaining how Tenisha got 1/8 again. Tenisha repeats her explanation.

Tenisha: Okay. First, I divided those two (pointing to the upper right side of the big rectangle) in half, and those two (pointing to the lower left side of the big rectangle) in half, and I counted all up which I got one-eighth.
Teacher: What kind of parts did you divide the whole into?
Tenisha: I divide this one (pointing to the upper right side of the big rectangle) and this one (pointing to the lower left side of the big rectangle).
Teacher: What kinds of parts did she make? Can someone use the word, that's really important here? What kinds of parts are we trying to make in the whole? Ahmed, what kinds of parts are we trying to make in the whole?
Ahmed: Rectangle parts.
Teacher: Rectangle, but there's very important word that is absolutely-
Ahmed: Equal parts.
Teacher: Equal parts. Thank you! Did she make equal parts?
Ahmed: Yeah.

In the same way, Tenisha points out the parts that she divided the rectangle into, describes that she counts all the parts, and gets 1/8. Tenisha's language choice "divided those two in half" is not sufficient in proving that all the parts are equally divided. After Tenisha's explanation, the teacher asks what kind of parts she made but Tenisha points to the parts that she divides into. Ahmed first answers the shape of part ("rectangle parts") but revises it into "equal parts." The teacher continues to elicit comments. Renee, who wrote 2/8 in her notebook, provides a comment. Unlike Tenisha, Renee drew diagonal lines in a way that all the parts have the same shape and wrote 2/8 for the green rectangle.

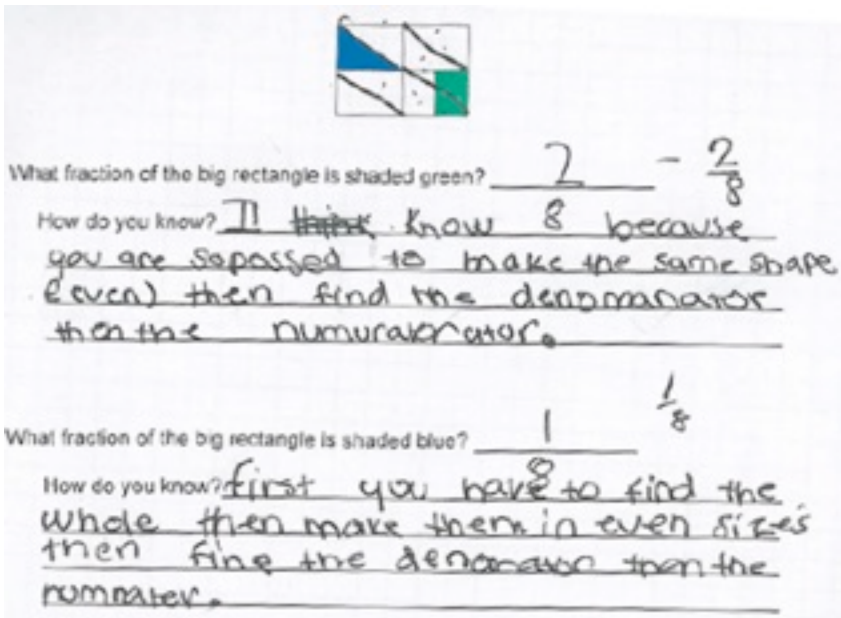


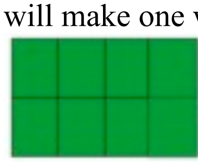
Figure 5.14. Renee's written record for the blue and green rectangle problem.

Renee:	Uhm, I have a question. Uhm, why-why did you like cut-like uhm, like, like divide those like their own things, but didn't divide the top one?
Tenisha:	(pointing the blue triangle part) This part?
Renee:	Uh-huh.
Tenisha:	Well, it seems like it's already divided just in a different shape.
Teacher:	Do you want to say something more, Renee?
Renee:	But they are not even. The shapes are not even-equal.
Tenisha:	Sometimes it doesn't have to be even to be…
Teacher:	Are you saying that these aren't equal parts too? Are these equal parts (pointing to the blue triangle and the small rectangle that is below the blue triangle) too? Are these two equal?
Tenisha:	[inaudible]

Teacher: I think what Renee is asking you how do you know that this right here (pointing to the triangle) is equal to this (pointing to the small rectangle)? Do you know that? Is that what you are asking, Renee?
Renee: Uh-huh.
Teacher: That's a very good question. Maybe somebody would like to come up and try sticking all of the pieces on and see if actually Tenisha is right, if they really are eight equal parts?

Despite Renee's vague description of "divide those like their own things" and vague reference of "the top one," Tenisha accurately points out the part that Renee refers to. Tenisha divides two small rectangles into the same shape as the green rectangle but does not draw a line over the blue triangle. Tenisha's method of proving still needs further justification that the blue triangle is the same size as the green rectangle. Tenisha initially responds that it is already divided into a different shape, but defends that it does not have to be even after Renee further challenges her that the shapes are not even. Tenisha defends her position of not dividing the small triangle that circumscribes the blue triangle into two little rectangles. It is not clear that the object of "does not have to be even" means "equal shape" or "equal size", but her justfication opposes the position of making the parts equal. At this point, the teacher asks Tenisha whether she thinks that the blue triangle and the small rectangle which is congruent to the green rectangle are equal parts or not, and then supports the students to prove using the sticky pieces. Connor comes to the board and covers the whole with the sticky green rectangles. While he is covering the whole with sticky green rectangles, the teacher explains that if it takes eight greens to cover the whole it proves that Tenisha is right. While Connor is covering the whole with green rectangles, other students count by 1/8.

Connor: I figured out that all these green stickies-green sticky pad things will make one whole rectangle thing-



Teacher: How many eighths up there now?
Connor: Eight.
Teacher: Can you count them all? By eighth? One-eighth
Connor and students: One-eighth, two-eighths, three-eighths, four-eighths, five-eighths, six-eighths, seven-eighths, eight-eighths.
Teacher: So what Tenisha said that one of them is what?

Connor:	One-eighth.
Teacher:	What did she say that one of them was?
Connor:	One-eighth.
Teacher:	And what Connor just showed is what? How many eighths does it take to cover the whole shape?
Student:	Eight.
Teacher:	Eight-eighths.
Connor:	One whole thing.

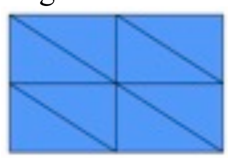
The explanation requires the number of green stickies to cover the whole, but Connor misses that information. The teacher asks the number of eighth in the drawing and counts by one-eighth. By showing that it takes eight of one-eighth to cover the whole, the class proves that Tenisha is right to call one of those equal parts is one-eighth. As the second session is almost over for Day 3, the teacher asks students to look back at what they wrote for the blue triangle and to see if they want to change their answer for the blue triangle.

On the next day, during the first session of Day 4, the teacher revisits the blue and green rectangle problem by recalling what Connor did with the green pieces.

Teacher:	What did Connor do with all of these green pieces? Here it is. Who can explain what he did? Demonte, what did he do?
Demonte:	Put them in equal parts
Teacher:	Speak louder.
Demonte:	Put them in equal parts.
Teacher:	He showed that you could make eight equal parts out of the whole and what's the name of one of those parts? What's the name of one of these parts, if there are eight equal parts? Aryanna?
Aryanna:	One-eighth.
Teacher:	One-eighth. This is one-eighth, this is another one-eighth, another one-eighth, another one-eighth, another one-eighth, another one-eighth, another one-eighth, another one-eighth. How many eighths are there in that whole? Connor?
Connor:	Eight.
Teacher:	Eight-eighths. And yesterday we said when you have eight-eighths, it's the same amount as the whole. We didn't figure out about the blue yet though. So we wanna figure out what fraction of that rectangle is shaded blue and we're going to ignore the green right now. And Renee asked very good question yesterday about when Tenisha said, remember Tenisha said she didn't have to make a line here because this is also two equal parts. You made lines here (pointing to the upper right part) and here (pointing to the lower left part). So now we're gonna just pay attention to the blue.

In reviewing the work that Connor did for covering the whole with eight greens, Demonte explains about making equal parts. With the support of the teacher's probes, Aryanna adds that one of eight equal parts is called one-eighth and Connor adds that there are eight-eighths in the whole. After a brief review about the green rectangle, the teacher moves onto the blue triangle. In solving the blue and green rectangle problem, some students have difficulty dealing with the other color, so the teacher reminds them that they only focus on the blue part and not the green one for naming a fraction that is shaded in blue. The teacher gives Liberty a turn to explain how many the blue triangles it would take to cover the whole.

Liberty: I figured out that-

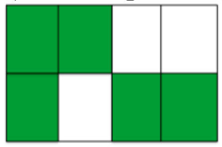


Teacher: Listen to Liberty please.
Liberty: It takes eight uhm triangles to make this whole.
Teacher: So it will take eight of those to cover it. How much is one of them? What fraction is just one of them, if you take eight of them to fill up? There're eight equal parts, then what would be called one of them? Renee?
Renee: One-eighth.
Teacher: One-eighth. Do you agree with that, Liberty?
Liberty: Yes.
Teacher: Let's count the eighth. Can we do that in the way what we did yesterday? Can you lead us? One-eighth?
Liberty and students: Two-eighths, three-eighths, four-eighths, five-eighths, six-eighths, seven-eighths, eight-eighths.

After Liberty provides an explanation that it takes eight triangles to make the whole, Renee identifies that one of them is called 1/8 and the teacher checks that there are 8/8 in the whole. The teacher then asks someone to come to the board to make 5/8 by removing some of the greens. Tenisha removes three green pieces.

Teacher: Who can figure out what to take off so that says five-eighths instead of eight-eighths. Thank you, Liberty. Let's see. How about Tenisha? Can you remove some so that says five-eighths instead of eight-eighths?

Tenisha: (removing three greens)



Teacher: Okay, can you explain how that's five-eighths? Listen to-listen to Tenisha's explanation.
Tenisha: It's five-eights-
Teacher: Talk a louder Tenisha.
Tenisha: Okay. It's five-eighths because three plus two is five if you take away three will be equal, oh no, never mind. I got confused.
Teacher: Can someone look at-you did a good job of making-you did correctly but who can explain, who can give a good fraction explanation of why that's five-eighths? Okay, Otis?
Otis: Because five green rectangles cover it and there's eight equal parts.
Teacher: And how much is one of these? What's the name of one of them?
Otis: One-eighth.
Teacher: Can everyone count up eighth? One-eighth
Students: Two-eighths, three-eighths, four-eighths, and five-eighths.

Tenisha correctly shows five-eighths but has difficulty providing an explanation for 5/8. Otis supplements Tenisha's explanation. The teacher then goes back to the original question and reviews the explanation for the blue triangle and the green rectangle. Ty explains:

> I think that it's one-eighth because it takes eight of those small green rectangles to fill the entire uhm, the entire large rectangle, so since, since one of rectangles is out of eight, it would be one-eighth.

With the teacher's request to repeat, Ty repeats:

> Okay, uhm, the rectangle (pointing to the green rectangle) shows-it will take eight of these small green rectangles to fill this entire rectangle (gestures to cover the big rectangle with his hands) and since there is one, that I could-you could kind of say, a kind of selected out of the entire rectangle, it would be one-eighth.

After Ty's explanation, the teacher checks the steps for naming a fraction written on the board. The teacher reviews an explanation for the blue triangle and Kalvin provides an explanation.

Kalvin: Because if you got those stickers and you put one right here, right here, right here, right here, right here, right here, and right here,

	that-it would make eight total triangles and… but like before like, there's only one triangle so it will be one-eighth.
Teacher:	Okay, who would like to comment on Kalvin's explanation? Did he do a good job with showing that it's one-eighth? What's your comment about it? How about Mark? What do you think about his explanation?
Mark:	I think that he was right because he actually explained clearly like Ty it is one-eighth, just the same as green shaded green, so I do agree with his.
Teacher:	Does anybody disagree with what Kalvin said?
Students:	(no one raises their hands)

After the teacher supports students in proving affirmatively that the blue triangle is one-eighth of the whole and the green rectangle is one-eighth of the whole, the teacher further challenges students by asking whether they are the same area or not.

Teacher:	Now, I have one more really hard question for you. Ready? So, here's what we figured it out. The green is one-eighth of the whole and the blue is one-eighth of the whole. That's the same fraction. Right? Is the blue right here, like this, is that the same size, not the same shape, I know it's not the same shape, but if this (pointing to the blue triangle) is one-eighth of the whole, this (pointing to the green rectangle) is also one-eighth of the whole, are they the same area or are they not the same area? They are the same number, but they do look very different. And I think Renee was a kind of asking about that yesterday. Who will be-who wants to say give us a reason why you think they are the same area and then we will see if someone wants to say they're not. Who would like to say they're the same? Okay, Kallie, what do you think?
Kallie:	I realized yesterday when I was looking at them, they didn't look the same. But what I was thinking that this is not for sure what I know but what I was thinking is I was thinking that blue and green are equal that they are not the same shape.

Before proving, the teacher collects several ways to prove it. Kallie suggests measuring with a ruler. The teacher does not test Kallie's idea right away, but continues to elicit another method for proving. Ahmed suggests a second method for proving.

Ahmed:	You can put all the rectangles across there and then you could get another chart and put all the rectangles across there, so if you put all the rectangles across there, that means it has the same area.
Teacher:	Okay. So I think-
Ahmed:	-cause you can-

Teacher: -you-

Ahmed: -cause you can fit them all in that whole.

Teacher: The same shape. So, why don’t we have of the-do we want to see if we can put all of the shapes on the same whole? Is that what you want to do? I think that Ahmed is-who can say what they think Ahmed is saying about all these shapes and all these shapes. How could we prove Kallie’s idea and what Ahmed said? Do you want me say one more time? Say one more time. It’s an interesting idea. Make sure everyone knows what you are saying. Look at him, please.

Ahmed: It’s like you-you can already fill the whole-the whole rectangle-the whole square with rectangles and triangles-

Teacher: so you can-

Ahmed: -but not at the same time.

Teacher: This is the same thing. It just doesn’t have lines in it. Who can explain what he is saying we can do with the shapes? Ty, what does he say?

Ty: I think he’s saying like to uhm, to put uhm also blue triangles with green triangles and make like half and half and to see altogether if they are equal area.

Teacher: Did you think half of the area or did you mean fill up the whole shape?

Ahmed: It’s like those two.

Teacher: Okay, maybe you can just say from these two. What are you seeing from these two?

Ahmed: They can both fill the whole.

Teacher: So what are you saying is that the blue-eight of the blue triangles fill up the whole and eight of the green rectangles fill up the whole.

Ahmed: That’s what I’m saying.

Teacher: So that proves what to you? What does that prove to you, Ahmed?

Ahmed: They are equal areas.

Teacher: So he says equal area cause it’s the same rectangle and both are filling up the same-eight of them fill take to fill up the shape.

Teacher: So he says equal area cause it’s the same rectangle and both are filling up the same-eight of them fill take to fill up the shape.

Ahmed’s suggestion is basically the same method that the class used to prove that each shape is one-eighth of the whole. The teacher restates Ahmed’s suggestion and then hints another way of proving by cutting the shapes.

Aryanna: You could cut them diagonal and…

Teacher: Cut this one (showing the green rectangle) diagonally?

Aryanna: Like a half

Teacher:	What if we overlap like this? Anyone have an idea how to do it? So, here's the blue one and here's the green one. (overlapping the green rectangle over the blue triangle)

Teacher:	Is there any way to cut them (peeling off the green rectangle and puts it next to the blue triangle) so that they turn out-so either the blue turns out to look like the green or the green turns out to look like the blue? Anybody have an idea how to try that?

Teacher:	Otis, you do?
Otis:	[inaudible] you could cut off the remaining edges
Teacher:	Cut off what?
Otis:	The remaining-like the edges your remaining parts of it.

Otis comes to the board and cuts the part of green rectangle and transforms it into the green triangle.

Otis:	Uhm, this is already on here, so I will cut the edge off and stick it back on here and the remaining parts would be sticked back on here for the edge right here.
Teacher:	Just stand back for a second and ask people what they think about that.
Otis:	What do you think about that?
Student:	It's cool!
Teacher:	Tina?
Tina:	I think that was very creative what he did right there because he got it out of the blue and cut the edge off and put it on the blue edge.
Teacher:	Does that prove that is the same amount of paper? Okay.

The teacher asks students to write in their notebooks whether they are convinced with Ahmed's explanation or Otis's explanation and wraps up the first session of Day 4 to take a break.

Summary

The EML 2013 students propose one answer (1/8) for the green rectangle and propose one answer (1/8) for the blue triangle. The incorrect answers (e.g., 1/2 or 1/4) are neither proposed by the students nor introduced by the teacher. The proportion of the students who recorded the incorrect answers by not taking the intended whole (e.g., 1/4) is low. The presented method, which does not dissect the whole into the same shapes, is challenged by the students. In explaining the blue and green rectangle problem on Day 3 and Day 4, the ELM 2013 students initially use the language incorrectly, inaccurately, and inconsistently; grant incorrect geometric names; do not use the comparative language in describing different sizes of the rectangle; and do not build a connection between the numerical representation and the pictorial representation.

To support students' development of mathematical explanation for the blue and green rectangle problem, the teacher provides mathematical supports upon an individual student's request but does not address the issues in a public space; invites the students to the board that allows to clarify what is being explained and to build a connection between a verbal explanation and a pictorial representation; uses established knowledge (i.e., definition of fraction) to support students to explain; uses the cutouts to support the students in proving that each part has equal size and in identifying the whole; and replaces inaccurate language and fills in missing words so that the students can appropriate the accurate mathematical term.

5.7. Summary of the Chapter

In Chapter 5, I analyzed instructional interactions for teaching the blue and green rectangle problem managed by the same teacher, Ms. Ball, to five different cohorts of the EML students. As a series of developing the definition of fraction in different representation models (area model, set model, and number line model), the blue and green rectangle problem is introduced to the EML students, following the brown rectangle problem, to reinforce the concept of identifying the whole and to extend on the concept of making equal parts in a part-whole relationship of the area model. The established knowledge from the brown rectangle problem makes the EML students to have an easy access to the blue and green rectangle problem, but the method of making equal parts by drawing lines in the brown rectangle problem is not sufficient in resolving the issues of the blue and green rectangle problem. Naming a fraction for the blue triangle is transferable to naming a fraction for the green rectangle but having the same fractional name would not be sufficient to convince that the blue triangle and the green rectangle are equal because of the different shapes. Both the brown rectangle problem and the blue and green rectangle problem share the same mathematical goal of developing the definition of fraction. However, because of the established knowledge from the brown rectangle problem, the blue and green rectangle problem has more demand for students in using language accurately, in building a connection between numerical representation and pictorial representation, and in building a logical structure of building an explanation than the brown rectangle problem but has less demand in preserving incorrect answers and in repeating, revoicing, agreeing, and disagreeing compared to the brown rectangle problem.

Similar to the difficulties in explaining the brown rectangle problem, the EML students have similar difficulties in explaining the blue and green rectangle problem which include the following: not establishing mathematical grammar to describe the objects to be explained; using inaccurate language in which its intended meaning is different from the accepted mathematical definition; skipping the logical structure of naming a fraction or paying attention to the partial components of naming a fraction; losing the purpose and the focus of what is being explained; and not building correspondence between an answer, an explanation, and representations. In addition to these, the EML students have difficulties in handling “a fraction of a fraction” algebraically and geometrically, being influenced by the established knowledge (e.g., counting the little grids), excluding the irrelevant information (e.g., counting the blue triangle and the green rectangle together), and being influenced by the visual layout or intuition.

CHAPTER 6.
CASE 3:
DEVELOPING MATHEMATICAL EXPLANATION FOR THE TWO-COIN PROBLEM

6.1. Overview

In this chapter, I analyze instructional interactions managed by the same teacher, Ms. Ball, for teaching the two-coin problem across two years (EML 2010 and EML 2013). The two-coin problem is consisted of two parts: (1) finding the amount of two coins out of pennies, nickels, and dimes; and (2) proving the exhaustiveness of solutions. The two-coin problem involves organizing multiple solutions systematically and making sure about the exhaustiveness of listed solutions. These are important mathematical practices, both as a way of finding a mathematical structure and declaring the mathematical end point of the work. As briefly described in Chapter 3, the two-coin problem has been used with slight variations, but the mathematical demand remains the same across the two years.

6.2. The Case of EML 2010

Preview

In the EML 2010, the two-coin problem is introduced on Day 3. After a brief set-up of the two-coin problem, including reading the problem statement aloud and restating what the problem is asking, the students start to work on the problem with a partner. A bag of coins, including the same number of pennies, nickels, and dimes, is distributed for each student to use. During a whole-group discussion, the teacher elicits solutions one by one instead of taking up one individual student's complete list. In the process of eliciting solutions, the EML 2010 cohort discusses that a different order produces the same amount of coins. After eliciting six solutions (DD=20 cents, ND=15 cents, NN=10 cents, NP=6cents, PP=2 cents, and DP=11 cents), Thailee makes an assertion that there are no more solutions. To support Thailee's assertion, Anthony reorganizes the proposed solutions into PP, PD, PN, DD, DN, and NN. After reorganizing the six solutions, the teacher asks the students to prove how they are sure that they have found all of the solutions. Due to the time constraint, the teacher wraps up the discussion of the two-coin problem on Day 3.

On Day 4, the teacher revisits the two-coin problem by asking a similar problem from homework. After that, the teacher shares the method that Bernard used and asks other students to interpret each row of Bernard's table. The teacher then further asks students to prove how they are sure that they have found all of the solutions. Karl addresses the idea that each combination is only used once and Coretta elaborates on Karl's idea by adding that flipping around the order of coins produces the same amount of money. The teacher returns to the reorganized list proposed by Anthony and supports the students to elaborate the method of finding all combinations without duplication: (1) with one penny, there are three possibilities; (2) with one dime, there are two possibilities; and (3) for one nickel, there is one possibility left. The extensive detailed analysis of instructional interactions for teaching the two-coin problem in the EML 2010 is provided below.

Extensive Detailed Analysis

In the EML 2010, the two-coin problem is introduced in the second session of Day 3, being framed as a learning opportunity to find a way to prove how you have found all of the solutions. In launching the two-coin problem, the teacher introduces different types of problems: problem with one answer, problem with multiple but finite number of answers, problem with infinite number of answers, and problem with none existence of answers. The teacher hints that the two-coin problem has finite number of answers but does not provide further information about the specific number of answers. After framing the two-coin problem in this way, the teacher asks students to read aloud the problem statement written on the poster. The following problem statement is written on the board:

> I have pennies, nickels, and dimes in my pocket. If I pull out 2 coins, what amounts of money might I have? Prove that you have found all of the amount possible.

The teacher asks Ahmed to restate the two-coin problem in his own words. Ahmed restates the problem as finding out two coins that are pulled out from the pocket. After his restatement, the teacher adds additional information:

- what needs to be recorded in the notebook: both the kinds of coins and the amount of coins
- pulling out coins with replacement
- minimizing a distraction from non-mathematical contexts (e.g., not necessarily pulling out coins from the pocket)
- reminding the norms of using coins
- finding all the different ways that you can make
- figuring out a system to keep track of solutions

Each student receives a bag of coins, containing pennies, nickels, and dimes, and then starts to work with a partner to find a way to figure out all the different amounts. During a partner work, the teacher provides advices, comments, and clarifications, which is sometimes initiated by the teacher and sometimes requested by students regarding:

- understanding the problem

- figuring out a way to keep track of solutions
- recording both the type of coins and the amount of coins
- clarifying the number of coins to pull out (i.e., two coins) to prevent further struggles; otherwise some students who pull out either more than two coins or less than two coins might neither be able to engage in the core mathematical idea during partner work nor be prepared well to enter into a whole-group discussion.
- clarifying the equal conditions of using materials (e.g., whether all students have the same amount of coins in the bag)
- decentralizing or not jumping into peripheral issues (e.g., an issue of repeating the same amount over time or jumping into probability issues)
- eliminating or pointing out the duplicated amount
- finding out all the different possibilities
- tracing the emergence of the core mathematical issues that contribute to the development of mathematical explanation and spotting resources to boost up the discussion (i.e., who will address the core mathematical issues in a whole-group discussion)

Among these, the following five issues are worthy to notice because the mathematical work in handling these issues are substantial for the development of mathematical explanations during a whole-group discussion.

The first issue being addressed is the method of recording. During a partner work, the teacher deals with cases in which several students have a record of either only the types of coins or the amount of coins, but not both. For example, Michael asks whether he is supposed to write down ten cents or two nickels. Having the record of the amount of coins only makes it difficult to track the type of coins at a later point. Because there are many different ways to make 10 cents except two nickels—such as 10 pennies, 1 dime, or 5 pennies and 1 nickel, but all of these do not satisfy the conditions of the problem—, students need to specify the combination that makes 10 cents. On the other hand, having a record of the types of coins only does not provide a quick reference in checking the duplication of the amounts at a later point. Rather, recording the type of coins only is more likely to lead to produce nine possible pairs, which double-counts the same amount of coins but different order of pulling out (e.g., penny and nickel; nickel and penny) as different solutions that is unnecessary for this problem, so it makes the mathematical work more complicate both for the students and the teacher. Moreover, having the record of both the type of coins and the amount of the coins makes an easy access for students to provide an explanation for why the proposed solution is a correct one in a whole-group discussion. During a partner work, either being initiated by the teacher or requested by the students, the method of recording is addressed explicitly in a public space.

The second issue being addressed is the repetition of the same amount of coins produced. During a partner work, several students raise their concerns, even complaints, about repeating the same amount over time. In general, using manipulatives (i.e., coins) makes easy access for students to produce correct solutions, not being distracted by incorrect solutions which violate the conditions of the problem. At the same time, however, it is also likely to create an issue of repeating the same solution over and over. This is illustrated by two episodes: (1) interaction between Karina and the teacher and (2) interaction between Madeline and Chanika. While the teacher circulates the classroom, Karina addresses her issue of repetition.

> Karina: I keep getting the same ones!
> Teacher: Yeah, that's not the problem. The problem should be, find the different ways you can combine the two coins.
> Karina: Yeah, like I keep putting up two, and I get ten or six.
> Teacher: Maybe that's not a good way to work. Maybe you should look at them and try to think of the different possibilities. That would-might work better.

Although Karina closed her eyes to randomly pick up two coins from her desk, she was only able to produce two solutions repeatedly—ten cents and six cents. For almost four minutes, the repetition of these two solutions holds off Karina on making a progress for listing other solutions. In theoretical probability, the probability of pulling out six cents (penny and nickel) is the same as the probability of pulling out 15 cents (nickel and dime) if the number of pennies, nickels, and dimes are the same and if a student pulls out two coins simultaneously. However, in the experimental probability, the probability of pulling out six cents is not always the same as the probability of pulling out 15 cents so students may repeat some combinations more than others, especially for the limited and small number of trials. Faced with this issue, without diverging into probability issues, the teacher asks Karina to think about possible combinations, not just relying on the coins that she gets from her empirical trials.

The second episode is from the interaction between Madeline and Chanika. During this exchange, the teacher talks with another student, so the instructional support is not visible in this segment. However, this exchange illustrates some mathematical issues that might prevent students from producing the complete list of solutions, thus provides implications for the work of supporting students to develop mathematical explanation. During partner work, both Madeline and Chanika hold several coins in their hands and drop two coins from their hands.

Chanika:	Watch this! Watch this! (dropping two coins from her hand)
Madeline:	Okay, I'm gonna…
Chanika:	I got five plus one. Don't I already got that? (looking at her notebook) Yup.
Madeline:	I already got a bunch of them.
Chanika:	I already got five…
Madeline:	I got (closing her eyes and dropping one coin from her hand) this and (closing her eyes and dropping another coin from her hand) this. What is that? Six cents. I already got it. Oh my gosh! It was the same one as you. (without adding more coins, dropping two coins from her hand) Oh my gosh! We got six cents again! (without adding more coins, dropping two coins from her hand again) What the heck is going on? (without adding any coins, dropping the remaining three coins from her hand) You know what? I didn't pick up any dimes.

After Chanika got six cents (with one nickel and one penny), Madeline got six cents (with one nickel and one penny) in her three consecutive trials. After dropping the remaining three coins from her hand, however, Madeline realized that she did not put any dime in her hand. The teacher distributes a bag of coins, intentionally including the same number of pennies, nickels, and dimes so that students make easy access to the solutions. However, by choosing a subset of coins, Chanika and Madeline did not keep the control of the number of each coin that the teacher gave, thus they were not able to get other solutions including a dime at the beginning of trials. This provides insights about how mathematical resources (e.g., coins) might impact on the access to all of the possible solutions, which is a prerequisite for developing a good mathematical explanation for the two-coin problem.

The third issue being addressed is the arrangement of coins. During a partner work, Amani asks the teacher whether a different order of coins would be the same solution or not.

Amani:	penny, dime and dime, penny equal?
Teacher:	No. That's a good question, though. Will you bring that up when we discuss it?
Amani:	Uh-huh.
Teacher:	And why do you th- Okay, so- I shouldn't have answered so quickly. Why do you think I'm saying, "no, they are- we don't count them different." Can you think of why? That was a great question. Sorry, 'cause I shouldn't have answered you so quickly. Why do you think that they're not the same?
Amani:	(silent)
Zahara:	(raising her hand) I think it's because it's basically the same thing, just rearranged an order.
Teacher:	It's the same amount of money. But it's a really good question, 'cause sometimes in math we care a lot about them being in a different order. But this problem we don't 'cause we're only concerned with how much money it is. But that was a great question. You should always ask questions like that in math, about if it matters, if they're- oh, a different order. That's a great question. That makes sense?
Amani:	Uh-huh.

Facing up with Amani's question, the teacher provides a quick answer and then asks her to bring the question to a whole-group discussion. After that, the teacher immediately amends her quick response and probes why they do not count "penny and dime" and "dime and penny" as two different answers. Instead of Amani, Zahara, a partner of Amani, provides an explanation that they both have the same amount but just have a different order. Through these exchanges, the teacher spots possible resources (i.e., the mathematical idea that Amani raised and whether Amani could deliver that issue) to boost up a whole-group discussion.

The same issue is also addressed by Macaulay. During a partner work, Anthony shares with the teacher that he believes he has found all of the solutions. After hearing Anthony's explanation, along with the six solutions that he produced, Macaulay cuts in on the conversation to share his observation. Unlike the interrogative sentence that Amani chose, Macaulay comments that switching the order of two coins produces the same amount of money.

Macaulay:	You can also put them backwards so they equal the same thing.
Teacher:	Excellent. So why don't we count the backwards ones?
Macaulay:	The backwards ones? (silent for a few seconds)
Teacher:	But why don't we use them for this problem? Do you know why?
Macaulay:	Why?
Teacher:	You just told me why.
Macaulay:	Becau- because it gets the same thing but backwards.
Teacher:	Same what? Same? Amount of money.
Macaulay:	Yeah.
Teacher:	Excellent. Very good. That's a good- yeah. That's very good. But now you've got to prove that you're done. Make sure that you have all of them.

Facing up with Macaulay's comment, the teacher makes it explicit why they do not count backwards as different solutions and then encourages him to engage in proving how he has found all of the solutions.

The fourth issue being addressed is to prove that you have found all of the possible solutions. During a partner work, three students, Zahara, Karina, and Anthony, announce to the teacher that they have found all of the solutions. Zahara's announcement is rashly added at the end of Amani's question about whether the order matters or not to the teacher and the moment when the teacher walks to another student, so her announcement is not further pick up by the teacher. As seen in Zahara's notebook, Zahara first lists three solutions with two different coins and then lists three solutions with the same two coins, but her explanation is based on her empirical experiment.

Zahara:	I think we have all of them because every time you pick up something it's the same thing.

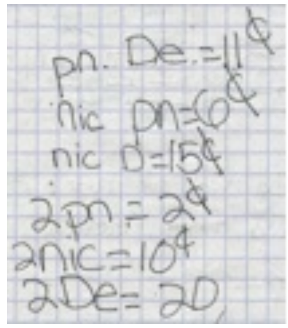


Figure 6.1. Zahara's notebook writing for the two-coin problem

While the teacher circulates the other side of the classroom, Karina makes an announcement that she has found all of the solutions. Because of the distance, her announcement is not heard by the teacher. In the notebook, she writes three solutions (nn, nd, and np), which is still an incomplete list of solutions.

Karina: I got all of them. There's only three you can do.

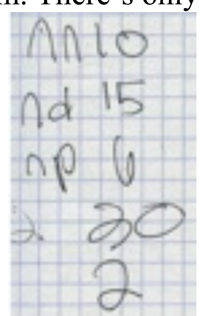


Figure 6.2. Karina's notebook writing for the two-coin problem

In contrast to the previous two cases of Zahara and Karina, Anthony's announcement is further expanded by the teacher.

Anthony: I believe I got all the answers.
Teacher: And do you think you have a way of proving that you have all of them?
Anthony: Because a dime goes with dime, it has two more ways. And I found two more ways of all of it. After they have nickel nickel, penny penny…
Teacher: There's two more ways with the- with the dime? Or what?
Anthony: The dime has two more ways.
Teacher: Okay, and then what? What did you do?
Anthony: Nickels, nickel, and then two nickels
Teacher: Okay.
Anthony: And then nickel and dime and nickel and penny.
Teacher: Okay.

Anthony: And two pennies, a penny and a dime, and a nickel and a penny.
Teacher: Very nice. Very nice. So why don't you write down how you think you know all of them. That's a very nice explanation, Anthony.

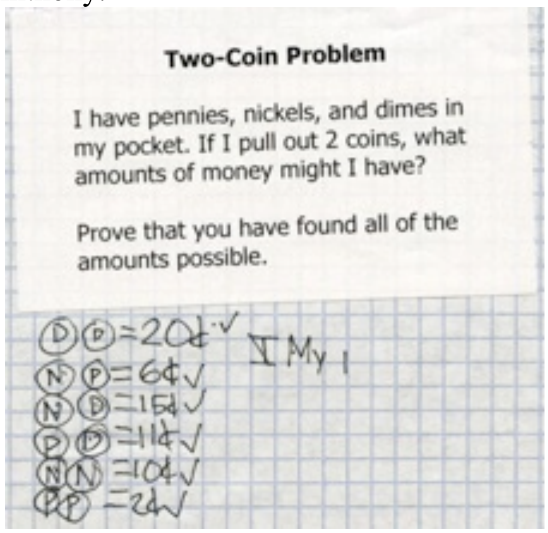


Figure 6.3. Anthony's notebook writing for the two-coin problem

Even though the systemic organization is not evident in the notebook, Anthony does systematically explain how he has found all of the solutions to the teacher. In his initial attempt, he outlined the idea that there are "two more ways" after the same coins. With the teacher's restatement of "two more ways with the dime," Anthony fixes the first coin (e.g., nickel), starts to add the same coin as a second one (e.g., nickel and nickel) and then expands to two other combinations with a different coin as a second one (e.g., nickel and dime; nickel and penny). He did not make an explicit about the duplicated ones, but it seems that making a reference to his notebook might be less problematic for the issue of duplication. Through this conversation, the teacher spots resources that she can use later to boost up a whole-group discussion.

In the notebooks, none of students have solutions that violate the conditions of the problem, but many of them have the issue of duplicated solutions[23]. The high frequency of duplicated solutions could be explained by the use of coins to solve the problem.

[23] No records are found in Macaulay's and Shelly's notebook.

Table 6.1. The EML 2010 students who produce all of six solutions

All of 6 solutions		Violation of the conditions		
		Yes, with the number of coins	Yes, with the type of coins	No
Repetition of the solutions	Yes, with the same order			Karl, Samara, Shar, Thailee
	Yes, with the different order			
	No			Anthony, Bernard, Chamaya, Coretta, Eric, Ella, Elias, Javonte, Jaclyn, Kassandra, Madeline, Michael, Terrence, Zahara

Table 6.2. The EML 2010 students who do not produce all of six solutions

Not all of 6 solutions		Violation of the conditions		
		Yes, with the number of coins	Yes, with the type of coins	No
Repetition of the solutions	Yes, with the same order			Jason, Ahmed
	Yes, with the different order			
	No			Dahlia, Hala, Karina, Mustafa, Qayshawn

After the 14-minute partner work, the teacher starts to collect solutions of the two-coin problem. In launching a whole-group discussion, the teacher neither asks the students to nominate the number of solutions nor picks up the complete list produced by an individual student. Instead, the teacher elicits solutions one by one. The first solution is shared by Anthony.

Anthony: Dime and dime.
Teacher: Okay.
Student: A penny and a penny.
Teacher: I'm just gonna write D for dime right now. A dime and a dime and how much money was that?
Anthony: Twenty cents.

Teacher:	Okay.
Student:	And a penny and penny.
Teacher:	Okay, somebody else? Okay, Terrence?

While Anthony shares his solution, one student interrupts to propose another solution. The teacher does not take up the solution which is not granted to make a proposal in a public space. Instead, she gives a turn to Terrence who raises his hand to get a chance to share his solution. Instead of inviting students to come up to the board to write their solutions on the board, which are usually considered a way of giving agency to students, the teacher controls the system of recording the proposed solutions on the board. In the above exchange, the teacher uses a representation of "D" to denote a dime and keeps the records of both the kind of coins and the amount of coins on the board.

Being authorized to share a solution, Terrence proposes a nickel and a dime. Whereas the teacher asks Anthony to explain the amount of the money that he proposed (20 cents), she now asks another student to supplement the amount of money that Terrence proposed (15 cents) and then checks with Terrence whether the amount is correct. In a similar vein, Dahlia proposes two nickels and Shar supplements the amount of money that Dahlia proposed (10 cents); Jason propose a nickel and a penny and Ella supplements the amount of money that Jason proposed (6 cents). Next, Macaulay proposes his solution.

Macaulay:	A D and a D.
Teacher:	Sorry?
Macaulay:	A D and a D. You can do it backwards.
Teacher:	(writing D D on the board) Okay, so what do you think about that? He's saying D and D backwards.
Student:	[…] the same thing.
Teacher:	Macaulay, you have your own answer to that already. So tell us what you figured out about that.
Macaulay:	I figured out if you have one D and one D, you can like switch them but it looks the same.
Teacher:	And you get what? The same-
Macaulay:	Answer. The same answer.
Teacher:	You get the same answer. So for this problem, what did you decide?
Macaulay:	I added- I thou- I thought that if I could do that I did it on each and every one.

Teacher: Right. And what did we find out? Since they're the same amount, what did you decide, Amani? Since they're the same amount when you flip them.

Amani: You can't- you can't do it 'cause it's the same.

Teacher: For this problem we're not doing that, but it's a very good question and several people asked me what if it's backwards? And in this problem because it asks how much amount of money, and you get the same amount, even if you get them in a different order, for this problem we don't have the ones that are the opposite order. (erasing "D D" on the board) But sometimes in math we would, so that's a really good question to ask if we care about reversing them. In this problem, we don't cause we're only counting the amounts of money that we get.

In proposing another solution, Macaulay is aware that his to-be-listed solution (dime and dime) is already recorded on the board but explains that his solution is "backward" of Anthony's. Because both coins are the same, it is not easy to detect how the Macaulay's solution (dime and dime) is backward of Anthony's solution (dime and dime), whereas the example that Amani's gave to the teacher during a partner work (penny and dime; dime and penny) is easier to figure out how they are backward. With Macaulay's proposal, the teacher first writes his solution on the board and then gives a turn to Macaulay again to unpack his thinking. Instead of immediately eliminating the duplicated answer that Macaulay proposes, the teacher uses it to address the issue whether the order matters or not. After Macaulay addresses that the backward one produces the same amount of money, the teacher gives a turn to Amani who raised the same issue during a partner work.

After erasing Macaulay's duplicated answer, the teacher continues eliciting other solutions. Hala proposes a penny and penny and supplements the amount of the coins (2 cents). Next, when Qayshawn proposes a dime and penny, the teacher asks Ahmed about the amount of the coins (11 cents) and then gets a confirmation from Qayshawn whether the amount is correct. As seen in these exchanges, the teacher sometimes asks the initial proposer to explain the amount of coins but sometimes she asks other students to supplement the amount of coins. Even after listing all of the six solutions, the teacher continues to give a turn to students who want to share their solutions.

Chanika: A dime and a nickel.
Teacher: A dime and what?
Chanika: A nickel.
Teacher: (writing Chanika's solution on the board)
Karina: We already have it.
Teacher: Karina? Go ahead. Say what you said.
Karina: We already have a nickel and a dime…
Teacher: And how much is it?
Karina: Fifteen.
Teacher: It's fifteen, and I-why are we not gonna use dime and nickel? Can someone explain why this problem we're not gonna write down dime and nickel? Does anybody know why? Coretta?
Coretta: Because we already have it written down.
Teacher: In what way do we have it written down? 'Cause we don't have this (pointing to Chanika 's solution) written down.
Coretta: We have a nickel and a dime, and all you did was switch it around.
Teacher: And they prod- how much money is that?
Coretta: It is fifteen cents.
Teacher: And both ways it's- and can you say both ways?
Coretta: Right.
Teacher: So, it's the same, so we are not gonna count it. Chanika, does that make sense? Yes? Okay.

When the teacher attempts to write the answer on the board, Karina jumps in and then claims that they already have it. As a reason for not including dime and nickel as another solution, Karina and Coretta explain that it is the same amount of money as with a nickel and a dime which is already listed on the board. After clarifying that the same amount of money is not counted again, the teacher goes back to Chanika to check that not including her proposal makes sense to her. The teacher continues to collect another proposal. Ahmed adds another proposal, a penny and a nickel.

Ahmed: Penny and a nickel.
Teacher: How much?
Ahmed: A penny and a nickel.
Teacher: Penny and a nickel?
Students: We have it.
Teacher: Where is it?
Karina: A nickel and a penny.
Teacher: How much money is that?
Ahmed: Six cents.
Teacher: Six cents.

When Ahmed proposes a penny and a nickel, several students jump in and explain that they already have it on the list. Similarly, Ahmed explains that his proposed solution has the same amount of money as a nickel and a penny that is already listed on the board. The teacher continues asking for another solution, but Thailee declares that there are no more solutions and Elias agrees. After Elias's agreement, Chanika proposes another solution.

Thailee:	There are no more.
Teacher:	There are- Thailee says there are no more. Elias?
Elias:	I agree.
Teacher:	You agree with that? Chanika?
Chanika:	Penny and a dime.
Teacher:	Penny and what?
Chanika:	A dime.
Madeline:	We already have that.
Teacher:	Penny and a dime. Where is penny and dime?
Students:	[…]
Student:	It's at the bottom.
Teacher:	Oh. It doesn't say penny and dime.
Student:	It says dime and penny.
Teacher:	So why am I not gonna write- why am I not gonna write penny dime?
Karina:	'Cause it's the same answer…
Teacher:	Say it again.
Karina:	'Cause it's the same answer just flipped around.
Teacher:	Do people agree with Karina?
Students:	Yes.

As soon as Chanika suggests a penny and a dime, several students claim that they already have it. After Karina's explanation, the teacher calls for the first agreement in this lesson. Here, the call for agreement is not merely to ignite the controversy over Karina's idea. Rather, it functions as checking the collective agreement on the established idea that switching around coins produces the same amount of money. This idea was already verified by Macaulay (his proposal for dime and dime after Anthony's dime and dime), Chanika (her proposal of a dime and a nickel after Terrence's proposal of a nickel and a dime), Ahmed (his proposal of a penny and a nickel after Jason's proposal of a nickel and a penny), and Chanika (her proposal of a penny and a dime after Quiachel's proposal of a dime and a penny).

After listing all of the six solutions (DD, ND, NN, NP, PP, and DP), checking that the switched-around solutions have the same amount (DN by Chanika, PN by Ahmed, and PD by Chanika), and no more the seventh proposal is made, the teacher opens up the discussion about how you are sure that you have all of the answers. Using her observation and interaction during a partner work, the teacher gives a turn to Anthony, who announced that he had found all of the solutions during a partner work, to initiate the discussion.

Anthony:	For example, if you have a penny and a penny, there's only two more answers. A penny and a dime and a penny and a nickel.
Teacher:	Okay. Penny- once you have two pennies you could do penny dime, you said, and then what could you do?
Anthony:	Penny nickel.
Teacher:	And what did you mean there're only two- after this there's only two more. What do you mean by that- is everyone listening to what Anthony is saying? Say it again, why do you think there's- after you do this there's only two more?
Anthony:	There's two more answers for penny.
Teacher:	For penny. Why?
Anthony:	[silent]
Teacher:	That's- what does the problem say? How many coins are you allowed to use?
Anthony:	Two.
Teacher:	Two, and at what points- which coins are you allowed to use?
Student:	Penny, nickel, dime.
Teacher:	Pennies-
Anthony:	Pennies, nickels, and dimes.
Teacher:	Okay. So once you use the penny with the other penny, then you're saying you can use it with the dime and-
Anthony:	Nickel.
Teacher:	And then what?
Anthony:	Nickel.
Teacher:	Are there any other ways to use a penny?
Anthony:	No.
Teacher:	Do people agree with Anthony? He's saying once you use the penny, they're only- with the other penny, there're only two more ways to use a penny, then you're done with the pennies. Is that right?
Students:	Uh-huh.

Similar to the explanation that Anthony gave to the teacher during a partner work, he systematically explains the organization of his solutions. After fixing a penny as the first coin, Anthony first matches with the same coin (a penny and a penny) and then matches with a different coin (a penny and a dime; a penny and a nickel). Even though the teacher is well aware of what Anthony means by “two more ways” for penny after a penny and a penny, based on her interaction with Anthony during a partner work, she calls for repeating the explanation in order to make the meaning more explicit to the class. After Anthony repeats his initial explanation, the teacher calls for the second agreement in this lesson and repeats his explanation again. The call for agreement functions as establishing the collective agreement on the exhaustiveness of solutions using a penny to make a progress to the next idea. The teacher continues to elicit an explanation from Anthony.

Teacher:	So then what did you do after that, Anthony? What ones- that’s only three, and we got six. So what did you do next after that to complete your answer? Did you try a different- did you work with a different coin next?
Anthony:	Yeah.
Teacher:	What did you work with?
Anthony:	Dime.
Teacher:	Okay. So you had dime, and what did you put with the dime?
Anthony:	A dime with a dime.
Teacher:	Okay, dime dime. And then what? Are there two more ways to use the dime?
Anthony:	Yes.
Teacher:	What are they?
Anthony:	[silent]
Teacher:	So what else can you do with the dime besides put it with that dime? With the other dime?
Anthony:	Nothing else.
Teacher:	Sorry?
Anthony:	Nothing else.
Teacher:	Just one? Just two dimes, that’s it?
Macaulay:	No, dime nickel… dime nickel.
Anthony:	A dime with a nickel.
Teacher:	Dime with a nickel.
Macaulay:	Then dime with a penny.
Teacher:	What about dime with a penny? What about dime with a penny?
Anthony:	We already have that.
Teacher:	We already have that. Do you see that? So what did you do next, Anthony?
Anthony:	Use a nickel.
Teacher:	Put the nickel, and what does the nickel go with?
Macaulay:	[The Dime.]

Anthony: [Nickel with the] dime.
Student: [You already that.]
Teacher: [You had that one here,] right?
Jaclyn: Is there another one?
Student: So another nickel.
Anthony: [Nickel with a nickel.]
Student: [Nickel nickel.]
Teacher: And what next?
Jaclyn: Nothing.
Student: Nothing else.

After listing the three possible solutions with a penny, Anthony starts with a dime as the first coin. Similar to his previous method, Anthony first matches with the same coin (dime and dime) and confidently answered that there are two more ways to use the dime except dime and dime. Despite of his affirmative reply, he could not list other two solutions further. Macaulay, who is a partner of Anthony, jumps in and lists a dime and nickel. Following Macaulay, Anthony adds a dime and a penny to the list. Macaulay, again, jumps in to list another solution of dime and penny, but the teacher directly addresses to Anthony about a dime and a penny. Anthony points out that it is already listed on the board. After Anthony puts a nickel, Macaulay again jumps in and suggests a nickel and dime. As Macaulay suggests, Anthony lists a nickel with a dime, but other students immediately respond that it is already on the list. For another solution, Anthony suggests a nickel with a nickel. The teacher continues to elicit other solutions, but a number of students assert that no further solutions exist. Through these exchanges, the original list is reorganized.

The original list			→	The reorganized list by Anthony	
D	D	20		P	P
N	D	15		P	D
N	N	10		P	N
N	P	6		D	D
P	P	2		D	N
D	P	11		N	N

Figure 6.4 The listed solutions written on the board for the two-coin problem

After reorganizing the solutions, the teacher goes on eliciting reasons why they have found all of the solutions.

Teacher: Why nothing?
Jaclyn: ‘Cause you already have…
Anthony: ‘Cause you have six answers.
Teacher: Jaclyn, what?
Jaclyn: ‘Cause you already have all of them, they’re just switched around.
Teacher: Can you show me the other places where we’ve used the nickel? We have nickel nickel, and what else do we have?
Jaclyn: Penny nickel, and dime nickel.
Teacher: Penny nickel and dime nickel. So Anthony, can you tell us why you think that shows you ha- you have all of them? I know you were thinking about that already. So why do you think that shows that there’s only six exactly?
Anthony: ‘Cause you use all the answers.
Teacher: So can you look up at what Anthony did? Does anyone have a question for Anthony? Anthony thinks this shows why we can be sure we have all of them. Can someone else explain Anthony’s thinking? Why does Anthony think this shows that we have every single one? Why does Anthony think this proves it? Qayshawn, are you listening right now? We’re trying to figure out why does Anthony think that this proves we have all the possible combinations? I only see two hands right now. Maybe that means people aren’t sure this shows that. Why does Anthony think this shows that? Zahara, what do you think? Why do you think this shows that? Why does Anthony think this shows that?
Zahara: Because you already used all the nickels, and if you used it again then it would just be repeating itself.
Teacher: Speak up a little bit, please.
Zahara: Because you already used all the nickels, and if you use them again then it’ll be- just be repeating itself.
Teacher: So once you’ve used all the nickels one way then you couldn’t use them and you would be repeating? Is that what you’re saying?
Zahara: I’m sorry. You can use all the coins. And then if you- like you can use how you did, and then if you did it more, then it would already just be repeating itself.
Teacher: Okay, so is that what you’re saying, Anthony? That if you’re using it this way and you started to write others, you would repeat?
Anthony: Yes.

After reorganizing all of the six solutions in a systematic way, the teacher continues to press students to explain why the seventh solution does not exist. Jaclyn proves it with the “switching around” fact and then adduces examples of two other solutions using a nickel except a nickel and a nickel. It is partially implicit, but basically based on the following logic: (1) For each coin, first match with the same coin; (2) After matching the same two coins, there are two more ways by matching the coin with a different coin; and (3) continuing to produce solutions with another coin, but eliminating the duplicates.

After Jaclyn's explanation, the teacher returns to Anthony. Unlike his earlier extensive explanation, Anthony provides a rather recursive explanation. The teacher asks other students whether they have a question for Anthony and then asks them to explain why Anthony's method proves that they have found all of the solutions. Instead of showing the exhaustiveness of answers, Zahara explains that further solutions would just repeat the-already-listed solutions. Because the time is almost running out at this moment, the teacher leaves space for students to think more about how they can be sure that they have found all of the solutions and then asks students to write down "Do you think we've found all of the solutions to the Two-Coin Problem? Explain your thinking." as the end-of-class check question. In the notebooks, several types of explanations are observed (see Table 6.3): empirical trials, repeating, using the conditions of the problem, matching coins, switching around, multiplication, recursive, and no further reason provided. Even after eliciting all of the solutions, it is not an easy task for students to provide an explanation about why they have found all of the solutions.

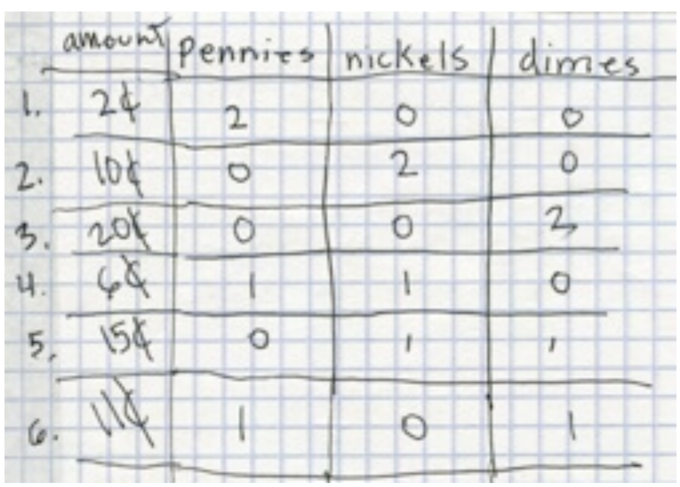
Table 6.3. Students' written explanations for whether they have found all of the solutions for the two-coin problem in the EML 2010

Type of explanations	Students' written explanations	
Empirical trials	Hala:	I think we did find all of the soloions because we went thouh two times and did not find any more.
	Michael:	Yes I think we found all the possible way because I did it as many times as I could.
Repeating	Chanika:	yes because if I took out the coin and keep dropping two am a get the same answers as what is on read on the board.
	Coretta:	yes, becaus it will keep repeting.
	Samara:	Yes I think we've found all of the coins because if we do some more, we will be repeting some.
	Zahara:	Yes I do because if you write any more than you might be reapeating your self
	Kassandra:	Yes, because I kept ~~po~~ pulling out the same combinations
Using the conditions of the problem	Ella:	yes because the is only ~~three~~ three coins.
	Qayshawn:	yes because we did not have 25¢.

	Madeline:	yes I think we found all the awnsers we would onley need more awnsers if we had Qurturs.
	Karina:	Yes because you car only bo it one time.
Matching coins	Devante:	Yes because we took all coins and matched them.
	Eric:	Yes, because when you use one coin and add it with the others coins, you do the same for the other two coins.
Switching around	Jaclyn:	Yes, because you can't swich it around and you use all the coins.
	Shar:	Yes, because you can't swich it around.
	Thailee:	Yes, Because you can't rewrite the same one over.
Multiplication	Bernard:	yes Because take The 3 (different coins) and the 2 (of How many coins you use) a multipliy Them together and you get Six.
Recursive	Anthony:	Yes I think so because we used all all the answers
	Ahmed:	I think Yes, Because we used them in all the ways.
	Terrence:	I think there are no more combonations because I have put them all together and and there are only 6.

	Macaulay:	Yec because ther was no more soulotions.
	Dahlia:	Yes because You cannot See no more of them.
No further reason provided	Elias:	Yes, I think we found all ~~th~~ the solutions.
	Karl:	yes I do think we found all the Solutions to the two coin problem.
	Mustafa:	yes we have found all of them
	Jason:	yes we did it was 5 of then.

In the second session of Day 4, the teacher revisits the two-coin problem to reassure that the students can prove that they have found all of the solutions. The teacher starts the session by reminding the similar problem from homework, using nickels, dimes, and quarters instead of pennies, nickels, and dimes. After a short reminder of a similar problem, the teacher projects Bernard's notebook on the board and then asks him to explain how he made a table and what it helps him with.



	amount	pennies	nickels	dimes
1.	2¢	2	0	0
2.	10¢	0	2	0
3.	20¢	0	0	2
4.	6¢	1	1	0
5.	15¢	0	1	1
6.	11¢	1	0	1

Figure 6.5. Bernard's notebook writing for the two-coin problem

Bernard: I made the table because the- with pennies, nickels, dimes, and the amount of how many- how much penn- the two coins would cost in the table, so I could write down, and I could check what I had made- what combinations I had made.

After Bernard's short introduction about his table, the teacher distributes turns to other students to see if they can interpret Bernard's table. For each row, one student interprets the amount of money in accompanied with the number of pennies, nickels, and dimes used. In every three rows, the teacher checks with Bernard to make sure that other students interpret his work correctly. By doing this, the teacher does not isolate Bernard while the audience is interpreting his work. After interpreting all of the rows in Bernard's table, the teacher asks if Bernard's table helps them be sure that they have found all of the answers. Javonte thinks that it helps him see using pennies, nickels, and dimes and Eric adds that it helps him see using exactly two coins in each row. Beyond the easy identification of the conditions of the problem, the teacher presses students to think further how they can be sure that there is not a seventh solution.

Teacher: So does anything up here help you be sure that there isn't a seventh answer? How do you know that somebody couldn't come in here and find a seventh answer to this problem. Somebody besides Coretta? So does that mean many of you think I could get someone to walk in here give us a- find another answer? How are you sure that you can't find anymore? Karl, what do you think?
Karl: Because we- you can only use one coin once, you can't use like dime and a nickel twice.
Teacher: Okay, so Karl's reminding us that we can't use like- if we use dime and nickel once, we can't flip it. Is that what you're thinking about?
Karl: Uh-huh.
Teacher: Okay. But how does that tell us we have six all together? Coretta?
Coretta: Because if you would try to flip it around, you would still come up with the same amount of money. And no matter how many times you flip it, you can only come up with one, because you only have pennies, nickels, and dimes, and you can't just keep on doing pennies, nickels, and dimes all over again, 'cause you keep getting the same answers that you got for the first time.
Teacher: Okay. So a lot of people think that's what- what Coretta said is right. That you keep getting the same answers again. Is that what a lot of you think?
Student: Yeah.

As an evidence to prove that there is not a seventh solution, Karl, as many other students wrote in the end-of-class check, explains that switching coins around is not counted. Without losing the focus of her original question, the teacher goes back to the question of proving all the solution spaces. In further probing about why it could be used as an evidence of six solutions, Coretta explains that flipping coins around produces the same amount of money. The idea of not counting the flipping around reduces the number of solutions from 9 to 6, but it is still insufficient to prove the exhaustiveness of the answers. In encountering the students' difficulties with seeing the exhaustiveness of the solutions, the teacher returns to Anthony's reorganized list that was made in the previous day and gives him a turn to re-explain his thinking. Anthony is pretty sure about the proof, but still not comfortable with coming up to the board to explain. He explains from his seat. Anthony explains the first three solutions on his list:

Anthony:	Once you used penny and penny, there's only two more options for- two more options. Penny-
Teacher:	So once you use penny-penny, there's two more options. What are they?
Anthony:	Penny and dime.
Teacher:	Penny and dime, and-
Anthony:	Penny and nickel.
Teacher:	Okay. Does everyone follow what he's saying? Once you put the penny down, there's only two more ways to use the penny. Does everyone ***agree*** with that?
Student:	Yes.
Teacher:	Could anybody walk into this room and find another way to use the penny?
Students:	No.
Teacher:	Why not? Ella?
Ella:	Because there's only three coins, and the only way you could find another one is by using another coin.
Teacher:	Okay, does everyone agree with that? There's only three coins, and the only way to make something different would have been to use another coin. Is that right? Okay.

In a consistent way with his previous explanation, Anthony starts by fixing the first coin and then goes through matching the second coin. He first matches with the same coin (penny and penny) and then introduces two other ways to match with different coins (penny and dime; penny and nickel). After eliciting all of the three cases with penny, the teacher seeks for a confirmation that there is no other way to match with a penny. The teacher calls for the collective agreement with the exhaustiveness of combinations with a penny and then moves to the next list with dime.

Teacher:	So then after you used pennies, you put down dime-dime and then what?
Anthony:	There's only one more way 'cause you already use dime and penny.
Teacher:	Can somebody repeat what Anthony just said? He said you put down dime-dime, and then there's only one more way because what? Who understood him, Jaclyn?
Jaclyn:	Because you already used dime-penny.
Teacher:	'Cause you already used dime-penny. Does everyone see dime-penny up here?
Student:	Yes.
Teacher:	Where is it? Where's dime-penny? I don't see it. Samara?
Samara:	Where it says eleven cents?
Teacher:	Over here, but we're looking at this row [pointing to Anthony's reorganized list] right now.
Samara:	Oh.
Teacher:	Let's not worry about this one [pointing to the original list]. Where's dime-penny in this one?
Samara:	Well, it's in a different order.
Teacher:	What is it?
Samara:	Penny-dime.
Teacher:	Penny-dime. And that's the same thing for this problem. We don't- in this problem we're interested in how much money it makes, so we don't count those different.

Following a dime and a dime initiated by the teacher, Anthony explains that there is only one more way to use with a dime because a dime and a penny is already listed on the board even though it is listed in a different order.

Teacher:	So then you put down what, Anthony?
Anthony:	A nickel and a nickel.
Teacher:	Okay, and then what?
Anthony:	And then there's no other answers, because you already used nickel and dime, and penny and nickel.
Teacher:	So you already used nickel and dime and penny and nickel, so there's nothing else to do with the nickels. Does everyone agree with that? So write down in your notebook if that- if what Anthony said convinces you that nobody could find a seventh answer. Or if you're not sure. So the questions is does that- does what Anthony said make you feel very sure that there couldn't possibly be a seventh answer no matter who tried? Or are you not

> sure. So you should either write, "I'm very sure" and tell why, or say, "I'm really still not sure." So we're- trying to figure out, are we sure there's not a seventh answer to this problem.

After checking the exhaustiveness of the solutions with a dime, Anthony goes through a solution with a nickel and explains that there are no more ways to use a nickel beyond a nickel and a nickel because a nickel and a dime and a penny and a nickel are already listed on the board. After asking for an agreement again, the teacher asks students to write whether they are convinced by Anthony's explanation. After exchanging whether the student are convinced about the exhaustiveness of answers, the teacher wraps up the discussion about the two-coin problem.

6.3. The Case of EML 2013

Preview

In the EML 2013, the two-coin problem is introduced on Day 4. Because the two-coin problem is assigned as a warm-up problem, the students start to work on the problem individually without any formal set-up of the problem. During the initial stage of the individual work, the requests are made by several students to help for understanding the problem. Facing up with these several requests, the teacher has a brief set-up of the problem in the whole-group setting: reading aloud the problem statement, restating what the problem is asking, eliciting an example from Anaya, and identifying the conditions of the problem. In along with, the teacher writes examples and non-examples on the board so that the students practice with using the conditions of the problem.

During a whole-group discussion, the teacher elicits solutions one by one instead of taking up one individual student's complete list (DN=15 cents by Kallie, PN=6 cents by Madeline, PD=11 cents by Aryanna, PP=2 cents by Tina, NN=10 cents by Connor, and DD=20 cents by Ahmed) and discusses whether the examples/non-examples written on the board meet the conditions of the problem. Jarvaise, Deshawn, and Demonte make proposals which violate the conditions of the problem, four pennies and ten nickels, two dimes and three nickels, and ten dimes respectively, but acknowledge that their proposals do not meet one of the conditions of the problem, namely using two coins. After no further proposals are made, the teacher asks students to write whether they are sure that they have found all of the solutions in the notebook and wraps up a discussion about the two-coin problem. The extensive detailed analysis of instructional interactions for teaching the two-coin problem in the EML 0213 is provided below.

Extensive Detailed Analysis

The two-coin problem is introduced on the first session of Day 4, being assigned as a warm-up problem. Like other warm-up problems, a formal set-up of the problem—a brief introduction about the problem, reading aloud the problem statement, and restating what the problem is asking—is not made in a whole-group setting. Instead, each student individually starts to work on the warm-up problem of Day 4 after doing daily routines (e.g., submitting homework, reading the teacher's comments in their notebooks, and pasting the warm-up problem on their notebooks). The following problem statement is written on the board:

> I have pennies, nickels, and dimes in my pocket. If I pull out 2 coins, what amounts of money might I have? Prove that you have found all of the amount possible.

During the initial stage of individual work, the following requests are made by students. The first request was made by Tina to clarify the number of coins in the pocket. There might be two possible reasons for this request. The first reason might be thought as seeking for the would-be-key information in the misinterpreted problem. If someone misinterprets the problem as figuring out the number of coins that are left in the pocket after pulling out two coins, the original number of coins in the pocket would be the key information for solving the problem. In another case, if someone misinterprets the problem as writing an equation with two types of coins (e.g., 6 pennies + 5 nickels = 31 cents), the original number of coins in the pocket would be the key information because it determines the maximum number of coins that could be pulled out. In both cases, the number of coins in the pocket would be the key but missing information to solve the problem, but, in fact, it is unnecessary information in the two-coin problem. The second reason might be related to concerns whether there are enough coins to pull out so that each type of coins could appear in more than one solution (e.g., if pulling out two pennies, is it possible to pull out another penny in the next solution?). An insufficient number of coins in the pocket[24] would damage the production of all possible solutions.

Facing up with Tina's request for clarification about the number of coins in the pocket, the teacher first assures that there are lots of coins in the pocket so no need to worry about that. Instead of delving into how Tina interprets the problem or why the number of coins in the pocket matters for her, the teacher eases Tina's concern and then explains to Tina that she can pull out two coins each time and then put them back again (i.e., pulling out coins with replacement). In doing so, the teacher prevents Tina from

[24] If someone pulls out two coins with a replacement, there should be at least two pennies, two nickels, and two dimes in the pocket. If someone pulls out two coins without a replacement, there should be at least four pennies, four nickels, and four dimes in the pocket.

misinterpreting the problem or from producing the insufficient number of solutions. In valuing her request for clarification, the teacher briefly checks with Tina whether a couple of coins might change the problem, but makes sure that there are enough number of coins in the pocket.

The second request is made by Kalvin to clarify what the problem is asking. Instead of providing a quick answer, the teacher makes sure that he reads the problem carefully enough and asks him to talk with partners about how they understand the problem. The same request is made by April. Neither of April nor her partner, Isabella, understands the problem, so the teacher asks April to read the problem aloud and then elicits an example: two pennies equals two cents. Without the substantial supports provided by the teacher but just reading aloud the problem by herself, April is able to produce one solution. After April writes down one solution in her notebook, the teacher encourages her to think about different pairs that she can pull out and write down both the type of coins and the amount of coins. After receiving the same request by Kalvin and April during the initial six-minute individual work, the teacher convenes the class to resolve their difficulties with understanding the problem. Given that no formal set-up of the problem was made and no materials were distributed to use, several students might have struggled with finding the entry point to the two-coin problem.

The teacher asks Ahmed to read the first part of the problem aloud ("I have pennies, nickels, and dimes in my pocket. If I pull out two coins, what amounts of money might I have?") but stops him from reading the second part of the problem ("Prove that you have found all of the amount possible"). After Ahmed's reading, the teacher asks Deshawn to restate what the problem is asking.

Deshawn: How much amount of money you have in your pocket.
Teacher: You skipped one step though. You don't just have it in your pocket, what do you do before-
Deshawn: -I mean uhm, it's still amount of how much pennies, nickels, and dimes that you add up with two coins.

Through Deshawn's restatement, one of misinterpretations is exposed: The two-coin problem is misinterpreted as simply finding out the amount of money in the pocket rather than finding out the amount of two coins pulled out from the pocket. If the problem is interpreted in this way, one needs the information about the number of coins in the pocket as Tina requested. After Deshawn rephrased the problem statement, the teacher elicits an example of the combination of two coins that could be pulled out from a bunch of pennies, nickels, and dimes.

Teacher:	Can someone give an example of one combination of two coins that you could pull out? So if there is a big bunch of pennies, nickels, and dimes in your pocket and you pull two coins. Who could give an example of two coins that you could pull out?
Anaya:	A nickel and penny.
Teacher:	A nickel and penny. Let's check the conditions of the problem. To be an answer, what does have to use? What coins? What coins you have to use to be a right answer? Yes, Demonte?
Demonte:	[silent]
Teacher:	What coins are possible in this problem?
Demonte:	Pennies, nickels, and dimes.
Teacher:	Pennies, nickels, and dimes. She said nickel and penny?
Demonte:	[inaudible]
Teacher:	So was that using the right kind of coins?
Student:	Yes.
Teacher:	Yes, that's the first condition. And how many coins does this problem say that you have to pull out? That's the second condition. Yes, Demonte.
Demonte:	Two.
Teacher:	Two coins. Did she pull two coins?
Students:	Yes.
Teacher:	Okay, so then it asks what amount of money is that? What amount of money is penny and nickel? Anaya?
Anaya:	Six cents.
Teacher:	Six cents. So the third thing you do is to write down the amount. So she would write one penny and one nickel, however she wants to write that down, she would write six cents. Then you have to find different pairs of two coins and find out how much money that is.

After eliciting an example from Anaya, the teacher extracts the conditions of the two-coin problem and then checks whether the example Anaya proposed meets the conditions. Because the class has been working on the conditions of the problem from the previous mathematical tasks, the term and the use of conditions are familiar for the students. The teacher underlines the conditions of the problem and numbers them on the poster: (1) pennies, nickels, and dime; (2) 2 coins; and (3) the amount of money. After checking up the amount of money of Anaya' example, the teacher asks Ahmed to finish up reading the rest part of the problem ("Prove that you have found all of the amounts possible.") By eliciting an example, the teacher identifies the conditions of the problem and makes an entry point to solve the two-coin problem. In separating reading the first part (i.e., finding all of the amounts) from the second part (i.e., proving the exhaustiveness of the list), the teacher gives an opportunity for students to thoroughly produce all of the solutions in the first part of the problem and prevents them from hastily jumping into the second part of the problem.

After the brief three-minute set-up in a whole group setting, the class resumes the individual work again. Even after eliciting an example and identifying three conditions of the problem, several requests are further made by the students to clarify some ambiguities that they have. Among the following four requests, the first two are quite similar to the previous requests by Tina, Kalvin, and April, whereas others are not.

- understanding of the problem
- clarifying equal number of pennies, nickels and dimes in the pocket
- clarifying the meaning of mathematical terms: proof and conditions
- preventing the further production of incorrect solutions that violate the conditions of the problem

First, even after a set-up of the problem in a whole group setting, including reading the problem aloud, eliciting an example, identifying the conditions of the problem, and checking whether the elicited example meets the conditions of the problem, two students, Jarvaise and Connor, express their difficulties with understanding what the problem is asking. When Jarvaise and Connor ask for help, the teacher reminds the example elicited from Anaya (a penny and a nickel equals six cents) and asks them to write it down in their notebooks. The example elicited from Anaya does not reduce the quality of mathematical work during an individual work but supports some students, who do not understand the problem, to access the staring point of the mathematical work, thus it makes an easy transition to produce other combinations with two coins. When just reading the problem over and over again does not help students understand the problem or materials (e.g., coins) are not available to use, the example makes an easy access to the solutions but does not decrease the mathematical demands of the two-coin problem.

Second, the request for clarifying the specific number of pennies, nickels, and dimes in the pocket is made by Otis. It sounds quite similar to Tina's request at the beginning of the class, but he seeks for the clarification in a more nuanced way.

Otis:	But what-then, if they were to do that, wouldn't they have more information, more specific about how much she has pennies, nickels, and dimes?
Teacher:	Tell me more about why you need to know that. That's a good question. Why do you think that you need to know that?
Otis:	Because she could have like way more pennies, nickels, and dimes and you could put more like-if she has more pennies, nickels, and dimes it has in, [inaudible] nickels, she can use two nickels or dimes or pennies.
Teacher:	You have enough, so you can use two or everything if you need to, but after you make penny and nickel like she did, you can't make penny and nickel again. So even though you have more nickels and pennies, you can't keep making the same thing cause you're trying to find all the different combinations. But you have plenty of pennies, nickels, and dimes, you don't have just a few. So I think-I think you're okay just start writing down all the combinations. That's a really good question though. Cause-what-maybe if I have given only one nickel in the pile, that would make it different right? But there's enough that you can make pairs. Okay! Good thinking, Otis!

Otis's request for the specific number of pennies, nickels, and dimes would be considered as seeking for the information that determines the maximum number of coins to pull out and influences on the number of same solution repeated. Similar to the response given to Tina, the teacher assures him that there is enough number of coins so no need to worry about that. The teacher again values the issue that Otis brought, but does not address this issue further in a whole-group.

Third, the request for clarifying the meaning of proof is made by several students. Without any previous exposure to the mathematical practice of proving during the EML, several students are not sure what it means to prove something and how to prove it. One issue with proving is related to the medium of proving. This includes whether the proof needs to be done through words (e.g., Anaya) or through listing all possible solutions (e.g., Otis).

Anaya: When it says prove that you have found all of the amounts possible, do you mean like prove right here or prove in words?
Teacher: How do you know that you're done? How do you know that there aren't any more?
Anaya: Okay.
Teacher: However you can think that you can prove it. And this isn't such a good proof all you say is like I can't think any more. That's not such a good proof. Cause maybe you just didn't think hard enough.

In responding to Anaya's question, the teacher translates the meaning of proving as how you make sure that the lists you produce are exhaustive and then illustrates an example of not a good proof.

Otis: When they say prove that you have found all of the amounts, does that mean you have to make a list all the amounts you can make with two coins with the money she has?
Teacher: And then you found-when you think you can't think of any more, try to figure out if you're really finished or maybe you forgot something. So if you can prove you found all the possible ones. So, first write down all the ones that you can find.

In this exchange, the teacher encourages Otis to think further about how he can be sure that he has finished with listing all the amounts. Another issues of proving is raised when several students jump to the second part of the two-coin problem.

Kallie: Dr. Ball? How would-I know that I'm not finished yet but how are we supposed to prove that-
Teacher: Why don't you first try to find them and then see if an idea occurs to you about how you can be sure you're finished?

In this exchange, the teacher assures Kallie to list all the possible answers first and then to think about a way of making sure how they know that they have found all of the possibilities.

Lastly, confronting the issue of producing incorrect solutions which violate the conditions of the problem, the further support is provided to practice the given conditions of the problem.

Kalvin: Dr. Ball, [inaudible] 20 dimes plus 10 pennies-
Teacher: -oh, no. What does it say? How many coins can you use?
Kalvin: Oh.

Teacher:	You missed one of the conditions of the problem. This is really- Kalvin, this is great like our number sentences problems, but this time, you can only use two. Do you understand that?
Kalvin:	Uh-huh.
Teacher:	That's why the conditions of the problem are so helpful. Cause that helps to remember.
Deshawn:	Excuse me, Dr. Ball.
Teacher:	Uh-huh.
Deshawn:	Can we use the same one like 4 dimes plus 4 dimes equals-
Teacher:	(to Deshawn) -no, you- (to Kalvin) Tell him why you can't do that.
Kalvin:	Do what?
Teacher:	(pointing to Deshawn's notebook) Why he can't do this?
Deshawn:	4 dimes plus 4 dimes equals something dimes?
Kalvin:	Because if you do that, then, cause 4 plus 4 that equal 80 dimes and 4 dimes plus 4 dimes would like what-
Teacher:	How many coins is that?
Kalvin:	That's two.
Teacher:	4 plus 4?
Kalvin:	Oh, no, that's 80.
Teacher:	And how many coins are you supposed to be using?
Kalvin:	Two.
Teacher:	Two. Where does the problem say two coins?
Deshawn:	Right here.
Teacher:	Yep. So, you can't use 8 coins.
Deshawn:	Okay.

Kalvin might misinterpret "two coins" as "two types of coins" and then produce an incorrect solution which violates the conditions of the problem. Making a directing reference to the conditions of the problem prevents the derailment from the solutions. Deshawn, Kalvin's partner, also produces an incorrect solution which violates the conditions of the problem, even though his intention is to clarify whether he could use the same coins more than once. After checking the conditions of the problem with the teacher, Kalvin is able to explain why Deshawn's proposed solution could not be accepted as an answer. Facing up with the incorrect solutions produced by two students, Kalvin and Deshawn, the teacher writes examples and non-examples on the board and asks students to practice whether these meet the conditions of the problem.

Teacher:	(writing the examples and non-examples on the board) If you've finished in your waiting, another thing you could do is see if you understand the conditions of the problem and check these three possible answers that somebody tried for this problem and decide

> which ones are actually correct answers for this problem and which ones are not and why. So if you're waiting, you can check to see which one of these are answers for this problem and which ones are not answers for this problem. Do it on your own. Don't talk about it with me right now.

1. 3 dimes = 30 cents 2. 1 penny and 1 nickel = 6 cents 3. 2 quarters = 50 cents

The examples and non-examples listed on the board are a good mix of (1) the violation of the number of coins allowed to use and (2) the violation of the type of coins allowed to use. In addition, the teacher utilizes the example that was proposed by Anaya so that it does not reduce the demand of producing all of the solutions further.

During individual work, two distinctive features are noticed. First, being responsive to the difficulties that students face with, the teacher elicits an example and uses examples/non-examples to practice with checking whether the proposed solution satisfies the conditions of the problem. Second, few students have an issue of duplicated solutions, but many students, especially who do not list all of the six solutions, produce solutions which violate the conditions of the problem. In the students' notebooks, several types of explanations are observed (see Table 6.6): using the conditions of the problem, empirical trial, switching around, finding a pattern, recursive, and starting with each coin.

Table 6.4. The EML 2013 students who produce all of six solutions

All of 6 solutions			Violation of the conditions		
			Yes		No
			For the number of coins	For the type of coins	
Repetition of the solutions	Yes	With the same order			
		With the different order			Renee**
	No		Deshawn	D'lon	Aziz Kadeem Tina Connor David* Ella* Anaya Aryanna Liberty Ty Kallie Mark Tenisha Otis

Notes: * crossed out the violation of the number of coins
** writing the amount incomplete

Table 6.5. The EML 2013 students who do not produce all of six solutions

Not all of 6 solutions		Violation of the conditions		
		Yes, with the number of coins	Yes, with the type of coins	No
Repetition of the solutions	Yes, with the same order			
	Yes, with the different order	Kalvin* (5)		Nicholas (5) Elysa (3)
	No	April (5) Ahmed (5) Demonte (2) Tashawnah (1) Jarvaise (1) Isabella* (5)		Madeline (3) Bria (3)

Note: * crossed out the violation of the number of coins

Table 6.6. Student's written explanations for whether they have found all of the solutions for the two-coin problem in the EML 2013

Type of explanations	Students' Written Explanations	
Using the conditions of the problem	Liberty:	I know that I got them all because if you pulled out any other coins it would be wrong.
	Anaya:	I know that I ~~have~~ found all the amounts because it said you can only use 2 coins. And you make a lot of combanation with only 2 coins. So I know I found all of them.
Empirical trial	Ella:	I think I found them all because, there is no more [illegible] that I can try.
Switching around	Otis:	I think I've found all of them because, since I can't switch any of around, then, I cant make any more.
Finding a pattern	Aziz:	I know that I found all combinations because you use each coin 3x.
Recursive	David:	I think I found all the different amounts with 5, 10, 25 because I used all the combanations I think
Starting with a each coin	Tina:	Because you already got the pennies with everything you got the nickel with everything and you got the dimes with everything. Also you can double it, it's still pulling out 2 coins.

		Aryanna:	I know I found all the amounts because if you can only ~~use the coins o~~ pull out 2 coins ~~you can only pull out for each can pull out the coin~~ (so if you) add the dime (can be any coin) with a ~~so Anth~~ another dime that could be one of the amouts you pull out of your pocket and if you add a dime with a nickle and penny that would be all the amount you could pull out of you pocke also this follows ~~the~~ All the conidtions in the deletions.
		Ty:	I know I got every combination I could ~~using~~ making 2 coins out of pennies, nickels, and dimes because I put all my numbers in groups that were ~~understandable~~ understandable to me. In my first group I put each combination I could that started with pennies. In my next group I all combinations that were left that started with nickels In my last group I put all the combinations left that started with dimes. In these groups I made sure that I used every combination.
Incomplete List	Sure	Ahmed(5):	I found all of them because I used all of them and in diffrent order
		Nicholas (5):	I got all the acounts You pull my pocket
		Renee(3):	I think I have found all of them because you can only use each number 2 times in each place.
		Elysa(3):	I have all amout becuse you can make back worksed

	Unsure	April(5):	I know I did not do all the answers but I did a lot of answer(15) so I can proberly think of more.
		Demonte(0):	It goes on Because I Can't Cont It all. But I Can count Some So I got. a lot But It is not all of It.

After the additional 15-mintue individual work, the teacher starts a whole-group discussion about the two-coin problem. Before students provide the solutions, the teacher explicitly addresses her expectation that the students explain how it meets the conditions of the problem written on the board. Kallie provides the first solution:

Kallie:	Uhm… dimes plus nickels equal fifteen.
Teacher:	One dime
Kallie:	One dime plus one nickel equals fifteen.
Teacher:	Now can you tell us how it meets the conditions of the problem?
Kallie:	It meets the conditions of the problem by it has a nickel in it and it doesn't have-it doesn't have more than two-
Teacher:	-your eyes-[inaudible] still writing, your eyes should be on the chart. If you are still writing, it's okay. Sorry.
Kallie:	It doesn't have more than two coins, it has a nickel and a penny in it, I mean a dime, and the amount is correct.
Teacher:	Does everyone ***agree*** with her explanation?
Students:	Yes.

After the teacher corrects "dimes" to "one dime," Kallie rephrases "dimes" to "one dime" and "nickels" to "one nickel." The teacher then asks how it meets the conditions of the problem and seeks for the collective agreement on the answer. Here, the call for agreement functions as a collective approval rather than aiming at fully engagement with disputing each other or a quick checking about the correctness of the answer. And the teacher moves onto eliciting the second solution:

Madeline:	One penny and one nickel.
Teacher:	And how much is that?
Madeline:	Six cents.

Teacher:	Can you tell us how it fits the conditions?
Madeline:	Because… it's two, two coins and I'm not using the same coin.
Teacher:	Okay.
Madeline:	I'm using two different ones.
Teacher:	Does it say that you have to use two different ones?
Madeline:	(silent)
Teacher:	Does it say that you have to use two different coins, Madeline?
Madeline:	No, but I-
Teacher:	-no, you did but you don't have to. Did you use the right kinds of coins?
Madeline:	(nodding her head)
Teacher:	What's the second condition?
Madeline:	(silent)
Teacher:	What's the second condition for the problem (pointing to the number two on the poster)?
Madeline:	Two coins.
Teacher:	Two coins. Did you use two coins?
Madeline:	(nodding her head)
Teacher:	And did you get the amount?
Madeline:	(nodding her head) Yes.
Teacher:	Okay, does everyone agree with Madeline's answer?
Students:	Yes.

The teacher clarifies that using different coins is not necessarily required for the problem, but checks whether the answer meets the conditions of the problem, and then seeks for the collective agreement on the answer.

Aryanna gives the third solution (one penny plus one dime is eleven cents) and the class checks whether it meets the conditions of the problem. She then asks the collective agreement on the answer. After eliciting three correct solutions, with checking whether each proposed solution meets the conditions of the problem, Jarvaise makes a proposal which violates one of the conditions of the problem.

Jarvaise:	Uhm… I will do the hardest one. Four pennies and ten nickels.
Teacher:	Four pennies and four nickels?
Jarvaise:	No, four pennies and ten nickels.
Teacher:	And ten nickels. Okay. And what's the amount?
Jarvaise:	Fifty cents-I mean fifty-four cents.
Teacher:	Fifty-four cents?
Jarvaise:	Yeah.
Teacher:	How did you get fifty-four cents?
Jarvaise:	Because I-I counted, I counted two nickels, and I added four pennies.

Teacher:	Okay. Can you tell us how it meets the conditions of the problem? Is everybody listening carefully to Jarvaise?
Student:	Yes.
Jarvaise:	They say-they say you can use pennies, nickels, and dimes. Right?
Teacher:	Did you use pennies, nickels, and dimes?
Jarvaise:	No. I just used pennies and nickels.
Teacher:	That’s okay. You don’t need to use all of them. But did you use the right kinds of coins?
Jarvaise:	Yeah.
Teacher:	Okay. What’s the second condition?
Jarvaise:	(shrug his shoulders)
Teacher:	Right up here. You can look.
Jarvaise:	Two coins and-
Teacher:	Did you use two coins?
Jarvaise:	No.
Teacher:	How many coins did you use?
Jarvaise:	Ten.
Teacher:	Use more than ten. You use ten plus four more.
Jarvaise:	Yeah.
Teacher:	So how many coins did you use?
Jarvaise:	Fifty-four?
Teacher:	You use four pennies, that’s four coins and ten more coins. What is four plus ten?
Jarvaise:	You said four plus ten?
Teacher:	Yeah. You have four plus ten or ten plus four. How much is that?
Jarvaise:	Fourteen?
Teacher:	Fourteen. So you used fourteen coins, but you are supposed to only use?
Jarvaise:	Two coins?
Teacher:	Two coins. You used too many coins. You have a great answer, that’s very cool answer, but you did use more than two coins. Did people understand why this isn’t an answer even though he is right that it equals to fifty-four cents? Can someone explain why it’s not an answer to this problem? Jarvaise just explained why it is not an answer. Can someone else explain what he said? David, why it is not an answer?
David:	Because you can only use two coins, you can’t use coins more than two times.
Teacher:	You can use coins more than once, but you can’t use more than two coins. Okay? So I’m gonna cross it out. Even though it does equal to fifty-four cents, it doesn’t fit the conditions of this problem.

In the notebook, except the example that Anaya proposed, Jarvaise produced all incorrect solutions which violate the number of coins that could be used. Among these, he proposed the largest amount he produced as he thought that it is the hardest one.

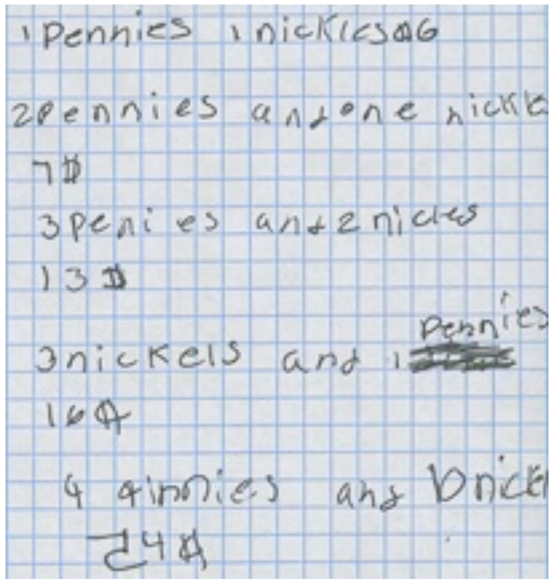


Figure 6.6. Jarvaise' written solutions in his notebook

Instead of rejecting the incorrect answer immediately, the teacher gives Jarvaise an opportunity to propose the amount of four pennies and ten nickels and how he got 54 cents. Jarvaise mainly explains the algebraic part, how he gets 54 cents by implying that two nickels are worth 10 cents thus 10 nickels are 50 cents and then add four pennies more. The teacher makes a direct reference to the conditions of the problem written on the board and Jarvaise does a self-check with the conditions of the problem. His proposed solution satisfies the first condition of the problem (using the right kind of coins) but does not satisfy the second condition of the problem (pulling out only two coins). Unlike responses to the other solutions, asking whether everyone agrees with the answer, the teacher asks other students to explain what Jarvaise said and why Jarvaise's proposed answer could not be accepted as an answer. David provides an explanation why Jarvaise's proposed answer could not be accepted as an answer for this problem.

Tina explains the fourth solution (a penny and a penny equals two cents), Connor explains the fifth solution (two nickels equals ten cents), and Ahmed explains the sixth solution (two dimes equals twenty cents). For each proposed answer, the teacher asks students to explain how it meets the conditions of the problem.

After eliciting all of the six solutions, the teacher moves to the examples/non-examples that she wrote during the individual work. Kadeem explains whether the first example on the board (3 dimes=30 cents) is a solution or not:

Kadeem:	3 dimes equals 30 cents. It's not one because it's using three and it says on the poster you're supposed to use two coins.
Teacher:	Okay, raise your hands if you agree with Kadeem's explanation. Can someone repeat what Kadeem said? How could you raise your hands that says you agree with if you don't know what to say? Kadeem, say that again. I want to everyone listen this time. Go ahead.
Kadeem:	3 dimes equals 30 cents.
Teacher:	Why is it not a solution?
Kadeem:	Because you're using three coins and it says on the poster that you have to use two instead of three.
Teacher:	Okay, who can explain what Kadeem said now? Elysa?
Elysa:	He said that it has three coins and it's supposed to have two coins.
Teacher:	You're only supposed to use two coins. So is this a solution?
Student:	Yes.
Teacher:	It is a solution?
Students:	No.
Teacher:	No. Even though it's the right answer, it doesn't fit to our problem.

After Kadeem gives an explanation why it is not an answer, the teacher seeks for the agreement with Kadeem's explanation and then asks other students to repeat his explanation. Through Elysa's repeat, the teacher makes explicit that the algebraic part is correct but it does not fit to the conditions of the problem.

Tashawnah explains how the second example (1 penny + 1 nickel = 6 cents) is a solution to the problem. Demonte explains why the third example (2 quarters = 50 cents) is not a solution for this problem.

Demonte:	We don't have quarters. Two quarters is fifty cents. We don't have quarters.
Teacher:	So, is it a solution?
Demonte:	No.
Teacher:	Okay, who can ***repeat*** what he said now? Otis?
Otis:	He said-he said, it-we don't have two quarters.
Teacher:	So? What does it mean that we don't have two quarters?
Otis:	We don't have two quarters. Up here, it doesn't say that you can use two quarters. Just no quarters up here. You don't-
Teacher:	No quarters in the problem. So, this one's not a solution either.

Demonte explains why the third example is not a solution to the problem and the teacher asks other students to repeat his explanation.

After checking all of the three examples and non-examples written on the board, the teacher continues to elicit another solution. Deshawn reports that his partner, Kalvin made another solution.

Deshawn: Uhm, Kalvin made [inaudible] two dimes plus three nickels equal seventeen cents.
Teacher: Two dimes and what?
Deshawn: Three-three nickels equal seventeen cents.
Teacher: Two dimes and three nickels?
Deshawn: Yeah.
Teacher: How much is that?
Deshawn: I said seventeen cents.
Teacher: Well, two dimes is twenty.
Deshawn: Uhm, that's twenty…
Teacher: Deshawn, do you wanna try to show how it fits the conditions of the problem?
Deshawn: Yes.
Teacher: Okay.
Deshawn: Uhm, it has uhm…
Teacher: What coins are you allowed to use? What does it say?
Deshawn: Two pennies, nickels, and dimes.
Teacher: Do you use the right kind of coins?
Deshawn: Yes.
Teacher: What's the second condition?
Deshawn: You have to use two coins.
Teacher: Did you use two coins?
Deshawn: No.
Teacher: What did you use?
Deshawn: Two dimes and five-
Teacher: -five coins-
Deshawn: -three nickels.
Teacher: So, is that a solution?
Deshawn: No.
Teacher: No. Cause you used too many coins. That's a kind of like, what Jarvaise was saying before.

Through checking the conditions of the problem, Deshawn is able to see that his proposed solution does not meet the conditions of the problem. Elysa suggests two dimes and Connor suggest two pennies, but they are both already listed on the board. Demonte suggests 10 dimes, but quickly admits that it does not meet the conditions of the problem. After repeating some of the solutions and proposing incorrect solutions which violate the conditions of the problem, the teacher asks students to write a complete sentence in the notebook whether they are sure that they have found all of the answers and wraps up the discussion about the two-coin problem.

Table 6.7. Students' written explanations for whether they have found all of the solutions for the two-coin problem in the EML 2013

Type of explanations		Students' Written Explanations	
Sure	Using the conditions of the problem	Anaya:	I think our class found all of them. Because, you can only use 2 ~~...~~ coins. And ~~so~~ I think that 6 is all you can find.
		Aryanna:	I think the class has found all the solutions, I think this because if you ~~do~~ want to follow the conditions this is all you can do for the answer.
		Mark:	We found 6 solutions i do agree with the class i think we found all the solutions because where only are supposed to use 2 coins.
		Liberty:	I know we found 6 Sloution The reason I know this is because if you pulled out any other it would be wrong
	Empirical trials	April:	We found 6 Slostens, I think are class found all the salosten, why becase first I most up and I only seen six one the board it's hard.
		Kalvin:	~~I thi~~ I think our class found all the soulutions because 10 of people in this class can't think of any more
	Recursive	Ella:	I think our class we have found all of the anwsers because there is no more answers we can't do.

	General belief about math	Kadeem:	I think our class has foand the solution because there are not alot of solution.
	Eliminating the incorrect answers	Nicholas:	I think we find all of them becase we hit the wrong anwers.
	Switching around	Tina:	~~think~~ I ~~know~~ our class got all of the possible amounts because it like yesterday how it fliped floped like But little differnt
	The limit of each coin used	Ty:	I think we found all the solutions because we used each coin 4 times, this is the limit
Not sure	No further reasons provided	Jarvaise:	I think are class have to work more
		Isabella:	~~I think ar~~ I think are class needs to fine more.
		D'lon:	I think our class has not foend all the solictnes

6.4. Summary of the Chapter

In Chapter 6, I analyzed instructional interactions managed by the same teacher, Ms. Ball, for teaching the two-coin problem to two different cohorts of the EML students. The two-coin problem involves producing multiple solutions that meet the conditions of the problem and proving the exhaustiveness of solutions. The mathematical explanation for the two-coin problem involves the following elements:

- Explaining why each proposed solution is acceptable as a correct answer for each proposal, which includes (1) using pennies, nickels, and dimes; (2) using only two coins; and (3) correctly adding the amount of the coins. Any violation of these three conditions could not be accepted as an answer for the two-coin problem.
- The repetition of the same kind of coin is allowed (at most two)
- Justifying that the order of coins does not matter (understanding what a different solution means)
- Grounding on the mathematical structure rather than empirical experiences

Building on the features of students' explanation provided at the end of individual year analysis in Chapter 6.2, this section provides a more general characterization and a more comprehensive collection of possible problems that individual students have in offering an explanation for the two-coin problem.

- Difficulties with accessing the entry point of the mathematical task: Lack of understanding about what the two-coin problem is asking is an obstacle to access to the entry point of the mathematical task. The possible misinterpretations lead to the request for unnecessary information (e.g., the number of coins left after pulling out two coins) and to the production of incorrect answers (e.g., writing a number sentence with two terms).
- The invalid transfer of prior mathematical knowledge (e.g., "writing number sentence for 10")
- Relying on the empirical experiences
- Lack of a system to build a structure of solutions
- Have less problems with inaccurate, incorrect, implicit, and vague language use

- What counts as an acceptable mathematical explanation for the exhaustiveness of multiple solutions

To support the students' development of mathematical explanation, the teacher provides the following instructional supports:

- Providing more extensive instructional supports at the beginning of the lesson, compared to the instructional supports provided for the brown rectangle problem (see Chapter 4) and the blue and green rectangle problem (see Chapter 5)
- Eliciting examples and non-examples to identify the conditions of the problem and to clarify the problem statement
- Eliminating incorrect answers at the beginning of the lesson
- Eliciting multiple solutions one by one instead of taking up one individual student's complete list
- Having more control over the way in which a public recording is made on the board

CHAPTER 7.
CASE 4:
DEVELOPING MATHEMATICAL EXPLANATION FOR THE THREE-PERMUTATION PROBLEM

7.1. Overview

In this chapter, I analyze instructional interactions managed by the same teacher, Ms. Ball, for teaching the three-permutation problem across three years (EML 2009, EML 2010, and EML 2013). Because the three-permutation problem is mainly assigned as a warm-up problem in the other two years (EML 2007 and EML 2008) without an extensive whole-group discussion, this chapter does not include the analysis of instructional interactions for teaching the three-permutation problem in those years. As briefly described in Chapter 3, the three-permutation problem has been used with slight variations in the instructional context (whether it is assigned as a warm-up problem or not), the instructional sequence of mathematical tasks (whether the two-coin problem is taught prior to the three-permutation problem), the number of mathematical tasks used, the context of the problem (three-car train, three-kids race, three-digit number), and the wording of the problem statement, but the mathematical demand remains the same across the three years. In the EML 2009, three versions of the three-permutation problem are introduced: the three-car train (red, light green, and purple) on Day 4, the three-kids race (James, Tasha, and Maria), and the three-car train (yellow, light green, and purple). In the EML 2010, two versions of the three-permutation problem are introduced: the three-digit number (1-2-3; 3-5-7) on Day 7 and the three-car train on Day 9 (yellow, light green, and purple). In the EML 2013, two versions of the three-permutation problem are introduced: the three-digit number (4-5-6) on Day 3 and the three-letters (a-b-c) on Day 5. Because the three-letters problem is mainly assigned as a warm-up problem without the extensive whole-group discussion, instructional interactions for teaching the three-letters problem is not included in this chapter. The problem contexts vary, but all of these three-permutation problems are mathematically isomorphic.

The three-permutation problem involves organizing multiple solutions systematically and making sure about the exhaustiveness of the listed solutions. These are important mathematical practices, both as a way of finding a mathematical structure and declaring the mathematical end point of the work. Developing a mathematical explanation for the three-permutation problem shares many features with developing a mathematical explanation for the two-coin problem. Major differences include the number of using the same element (at most twice for the two-coin problem; exactly once for the three-permutation problem) and whether the order of arrangements matters or not, (the order of arrangement does not matter for the two-coin problem, whereas the order of arrangement matters for the three-permutation problem) but both mathematical tasks require to explain the exhaustiveness of six solutions by finding the systematic organization. In this sense, it is worthwhile to attend to the instructional sequence of the two-coin problem and the three-permutation problem. The two-coin problem is not introduced in the EML 2009, is introduced before the three-permutation problem in the EML 2010, and introduced after the three-permutation problem in the EML 2013.

7.2. The Case of EML 2009

Preview

In the EML 2009, the three-permutation problem is introduced within different contexts three times: three-car train (red, light green, and purple) on Day 4, three-kids race on Day 5, and three-car train (yellow, light green, and purple) on Day 9. On Day 4, the three-car train (red, light green, and purple) is introduced as a warm-up problem. The requests made by students to understand the problem statement lead to eliciting examples (grp and rgp), identifying the conditions of the problem, and clarifying the meaning of different three-car train. The three-car train problem is mainly introduced as a warm-up problem, so the whole-group discussion is not extended to explaining the exhaustiveness of solutions.

On Day 5, the three-kids race problem is introduced as a warm-up problem. Instead of taking up one individual student's complete list, the teacher elicits solutions one by one: James, Tasha, and Maria by Marcellus; Tasha, James, and Maria by Alvan; Maria, Tasha, and James by Aiyana; Tasha, Maria, and James by Nina; James, Maria, and Tasha by Natania; and Maria, James, and Tasha by Ricky. After eliciting six solutions, Elina asserts that there are no more solutions and proceeds to pairing the listed solutions by those that have the same kid in the first place. Callie adds that each student has won the race at least twice.

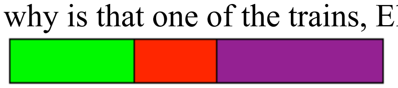
On Day 9, the three-car train problem (yellow, light green, and purple) is introduced as a warm-up problem. Instead of taking up one individual student's complete list, the teacher elicits solutions one by one: gyp by Mannis; gpy by Natania; ygp by Tiara; pgy by Evan; ypg by Tonya; and pyg by Levi. The exhaustiveness of solutions is proved by the ideas that the pair of two solutions is switching around. After discussing six solutions for the three-car train problem, the class shares their idea of how the three-kids race problem is similar to the three-car train problem.

Extensive Detailed Analysis

> Find all the ways to make different trains using exactly one of the red, light green, and purple rods. Keep track of each train. How many are there? How do you know you made all the possible trains using just those 3 rods?

The three-car train problem is introduced on the first session of Day 4, assigned as a warm-up problem. Like other warm-up problems, the students start to work on the warm-up problem individually without any formal set-up. A bag of Cuisenaire rods is distributed to individual students. The teacher makes sure that the students take out the right Cuisenaire rods from the bag. At the very beginning of individual work, Riya expresses her difficulties in understanding the problem to the teacher. Instead of providing an individual response to Riya, the teacher convenes the class to make sure that the students understand what the problem is asking. The teacher asks Alvan to read aloud the problem and elicits an example from Elina.

Elina: Green... red, and purple.
Teacher: Green, red, and what?
Elina: Purple.
Teacher: (building a "grp" train using Cuisenaire rods on the board) Okay, why is that one of the trains, Elina?



Elina: Because it uses- because it uses light green, red, and purple.
Teacher: [How many of each?]
Elina: [...red.] Just one.
Teacher: (pointing to "exactly" on the board)
Elina: Exactly.
Teacher: Exactly one each. So that's one of the trains that you don't have to add up how many people are on it, or anything. Just make the- build it. Or record it your- what is a way you could record this train in your notebook? What is a way to write down the name of the- what the train is? Callie?
Callie: You say the colors.
Teacher: What?
Callie: You say the- the beginning letter of the colors.
Teacher: Yeah, so what- what would you write for this train?
Callie: Uhm, G, R, and P.

Teacher: G, R, and P would be one way you could record this train. And you have other ways you could record too. But that's all you need to do is build trains that use exactly one of each of those rods, and record them. Does that help? Okay, so I'm gonna walk around. And I should see people trying to make the trains, and trying to find all the trains that are possible to make.

The teacher builds the arrangement that Elina proposed (grp) on the board and then asks why it is acceptable as an answer for the problem. To be accepted as an answer, it needs to meet two conditions of the problem: (1) use light green, red, and purple; and (2) use exactly once. Although the class does not discuss the term "condition" yet, the teacher makes explicit that the answer should meet the conditions of the problem. By eliciting an example, the teacher supports students to access to the entry point of the problem and identifies the conditions of the problem explicitly.

In addition, the teacher comments that students do not need to add up the size of trains. In doing so, the teacher help students not being confused with the Train Problem Part 1. The Train Problem Part 1 says:

The Train Problem

The EML Train Company makes five different-sized train cars: a 1-person car, a 2-person car, a 3-person car, a 4-person car, and a 5-person car. These cars can be connected to form trains that hold different numbers of people.

1-passenger car
2-passenger car
3-passenger car
4-passenger car
5-passenger car

Part 1

Try to build some trains. You can use only these five types of cars to build trains, and you can use at most one of each type of car in each train.

What are the different numbers of people that the EML Train Company can build trains to hold?

Although there are similarities in building a train between the Train Problem Part 1 and a three-car train problem, the conditions of the problem and mathematical goal are different across these two mathematical tasks. The main mathematical differences between the Train Problem Part 1 and the three-car train problem are as follow:

- The critical condition of the Train Problem Part 1 is to use at most one of each rod, whereas the critical condition of the three-car train problem is to use exactly one of each rod.
- The mathematical point of the Train Problem Part 1 is to figure out the length of the train, whereas the length of train is fixed in the three-car train Problem.
- The order of arrangement does not matter for the Train Problem Part 1, whereas the order of arrangement is the key idea for the three-car train problem.
- Different answer means different length of the train for the Train Problem Part 1, whereas different answer means different order of rods for the three-car train problem.

During individual work, several students wrote expressions or equations for the train they built. This is not mathematically problematic, but is unnecessary for the three-car train problem.

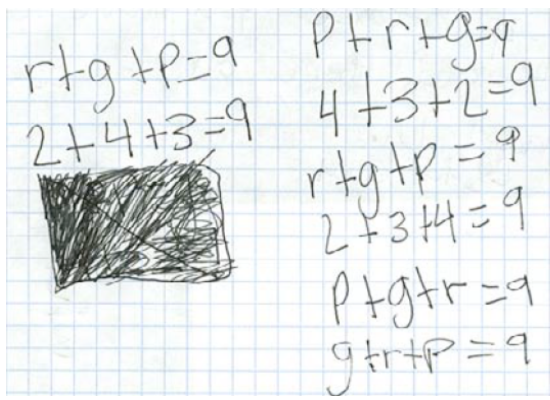


Figure 7.1. Aiyana's notebook writing for the three-car train problem

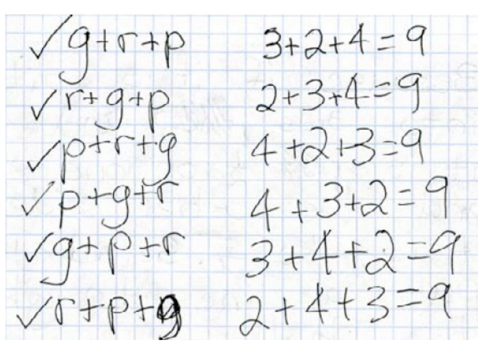


Figure 7.2. Elina's notebook writing for the three-car train problem

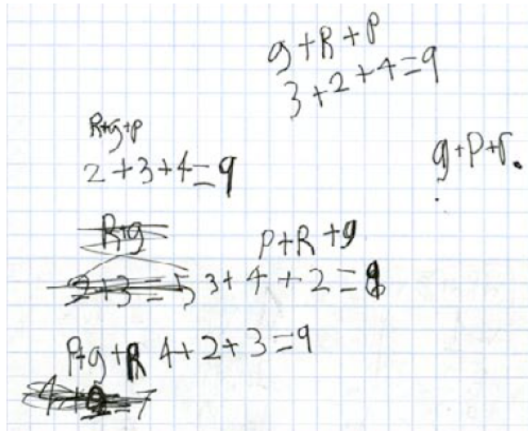


Figure 7.3. Ricky’s notebook writing for the three-car train problem

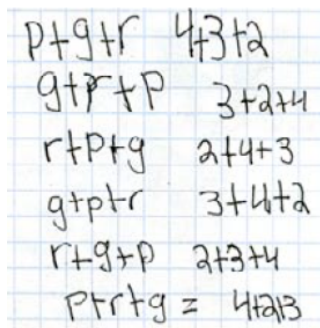


Figure 7.4. Amelia’s notebook writing for the three-car train problem

After identifying the conditions of the problem and eliciting one of the solutions, the class returns to their individual work. During individual work, Riya checks the number of each rod that she could use to build a train and addresses her concern that it makes the same train.

Riya:	Do you have to use one of each?
Teacher:	(pointing to the poster on the board) What does it say? Use what?
Riya:	Each one... Isn't this one the only one you can make?
Teacher:	What does it say in the problem? Make different trains using what?
Riya:	One of each?
Teacher:	What- what does it say before one of each? What word comes before the word one?
Riya:	Exactly.
Teacher:	Exactly one of each. So you have to use exactly one of each. Do you know what that means?
Riya:	So we can use two?
Teacher:	You can't use two, because it says exactly one.

Riya:	Well one would be the same problem using purple, red and green?
Teacher:	Right, but you can put them in a different order, right?
Riya:	Wouldn't it be- still the same problem?
Teacher:	So you're saying if I rearrange those it'll be just the same train again?
Riya:	No. It would be different.
Teacher:	That's what your question is though, right?
Riya:	Yes.
Teacher:	Because it'll be the same rods again.
Riya:	Uh-huh.
Teacher:	Do you wanna ask- do you want to ask the class? Or do you want me to just tell the class how to think about that? Do you want me to explain it?
Riya:	Yes.
Teacher:	Okay.

Riya double-checks with the teacher whether she is allowed to use exactly one of each rod and addresses her concern that following the conditions of the problem (using exactly one of each rod) produces only one solution. Influenced by the previous work on the Train Problem Part 1, in which different trains mean different size of the train, Riya might consider that she could only produce one solution with the given conditions. The teacher addresses this issue to the whole-group again.

Teacher:	So Riya just asked me a really good question that I think is important in this problem. She notices that we have to use exactly one of each of those colors. So, what would be another one I could make? Can someone give me a second one? That follows the one Elina told me? And then we could answer what Riya's bringing up. It's a very important question Riya's bringing up. What's a second train we could make? Mannis?
Mannis:	Red and purple.
Teacher:	Okay. Why red and purple?
Mannis:	Well that's another train you could use.
Teacher:	Say that again?
Mannis:	That's another train you could use. [And-]
Teacher:	[Why-] how do you know you could make that one? How does it fit the problem?
Mannis:	Because it's one of the trains you can use, and- well, it's one of the cars you can use and uhm… there's only one of each car.
Teacher:	What does this one say? (pointing to the poster on the board) How do you make the trains? You're making different trains using what?
Mannis:	The rods.

Teacher: Well [inaudible] different trains using what? What does it say?
Mannis: Exactly one of each, of the red.
Teacher: Okay.
Mannis: Light green, and purple.
Teacher: So is your- does your train use exactly one of the red, light green, and purple?
Mannis: No.
Teacher: What it- why not?
Mannis: Well, it doesn't- well- do you have to use all of them in each train?
Teacher: What do people think? Do you have to use all the colors in every train? That's I think the question we have to talk about. So you're- you've got your finger right on the right question, Mannis. What do you think, Callie?

To address the issue raised by Riya raised, the teacher elicits another example. Mannis proposes "red and purple" which does not meet one of the conditions of the problem, "exactly once." The teacher takes this moment to clarify what is acceptable as an answer for the three-car train problem.

Callie: I think that we should just- I think that you should just switch them around to get different kinds of trains different ways.
Teacher: Could people hear Callie?
Student: No.
Teacher: No.
Student: No.
Teacher: Say it again and but talk out to the class a little bit better.
Callie: I think that she means that we have to just switch him around to- to make different kinds of trains. But with the same colors.
Teacher: Okay, so Callie's interpretation is we're gonna be switching them around to make different ones, and the question is- is this one of the trains then? (pointing to the train that Mannis suggested: purple and red) This is a little bit switched around. Because it- purple is first. So is this one of the- is this the second train? Or not? It says, make different trains using exactly one of the red, light green, and purple. Is this using red, light green, and purple exactly one of each?
Students: No.
Teacher: So what- what do you think, Mannis?
Mannis: [...] put a light green at the end, and [make it a train].
Teacher: [You have to put a light green] at the end. (add "light green" rod at the end of train. The train built now is "prg" on the board) Why is that, Mannis?
Mannis: Because it's in a different form?

Teacher:	Okay, but why do you need to add the green on for what you originally told me?
Mannis:	Because you need one of each color.
Teacher:	You need exactly one of each color. Okay? So how would I record this one? Can someone tell me how to record it? Nina?
Nina:	P, R, G.
Teacher:	Okay, P, R, G.

Callie suggests to use the same colors as Elina's solution but just to arrange them differently. After hearing Callie's comment, Mannis revises his solution from "pr" to "prg." The teacher then asks whether the first solution proposed by Elina (grp) is the same as or different from the second solution proposed by Mannis (prg).

Tonya:	They're different.
Teacher:	Why are they different?
Tonya:	[...] different orders and- they just put in different orders.
Teacher:	They're in- they're in different orders. And for this problem, if it's in a different order, it's a different train. I think that's the question Riya wanted to know. Because it's the same colors again. But it's a different order. So what Callie said, can you- now can you say again what you said the problem is asking? What are we trying to do?
Callie:	We have to switch them around- and to see how many ways we can- we can make trains, but with the same blocks.
Teacher:	Okay, you're trying to switch them around and find how many different trains you can make using the same blocks but in different order. Does anyone have a question now? Okay, so see what you can build, and then we'll see if we can collect what you came up with in a few minutes. Okay?

After Tonya's explanation, the class resumes individual work again. After 12-minute individual work, the teacher asks students to confer with a partner and compare their lists, if both of them finish listing all of the solutions.

After 3-minute of partner work, the teacher convenes the class for a whole-group discussion. Instead of leading a discussion about proving how you know that you have found all of the solutions, the teacher makes clear that she does not want to lead a full discussion about the solutions because not all of the students have a chance to finish the problem. Instead, she highlights the term "exactly one" by contrasting two terms with "at least one" and "no more than one" which were used in other mathematical tasks.

On Day 5, the three-kid race problem is introduced as a warm-up problem.

Three kids ran a race. James, Tasha, and Maria. We don't know the results. We just know that one person finished first, someone finished second, and someone finished third. Make a list of all of the possible results of the race. How do you know that you found all of the possibilities?

During individual work, the request for understanding the problem statement is made by several students. When Malik addresses his difficulties with understanding the problem, the teacher asks him to confer with his partner, Ricky. After a few minutes, the teacher comes back to Malik to check whether he figured out what the problem is asking. Malik responds that he forgot to read one line and is able to figure out the problem after reading it aloud. As another example, when both Tiara and Jacqueline do not have a full understanding of the problem, the teacher asks them to read aloud the problem to each other and then discuss it with each other. Tthe teacher returns to Tiara and Jacqueline after a few minutes to check whether they have made progress in understanding the problem.

Teacher:	Is it- it is not making sense still? Is that making any more sense yet? Okay, so what's the story about?
Tiara:	A race.
Teacher:	Okay. What- and who's in the race?
Tiara:	James, Tasha, and Maria.
Teacher:	And- what happens when they run the race? What can happen?
Tiara:	Someone wins.
Teacher:	Someone wins, someone comes in...
Tiara:	First.
Teacher:	First.
Tiara:	Second, and third.
Teacher:	Okay, so what's one example of how the race could finish with those three kids? Some- who could be first?
Tiara:	Tasha.
Teacher:	Okay, Tasha could be first, who would be second?
Tiara:	Maria.
Teacher:	Who?
Tiara:	I think it's Maria.

Teacher:	Oh, Maria. Uh-huh.
Tiara:	Maria.
Teacher:	And who could be third?
Tiara:	James.
Teacher:	Okay, so why don't- can you write that down? Tasha- what did you say? Tasha...
Tiara:	Tasha.
Teacher:	Tasha, Maria, James.
Tiara:	(writing "Tasha, Maria, James" in her notebook)
Teacher:	Okay. So now can you think of a different way the race could finish? Somebody- a different way that's not exactly like that?
Tiara:	James, Maria, and Tasha.
Teacher:	(to Jacqueline) Did you hear what she said?
Jacqueline:	(shaking her head)
Teacher:	(to Jacqueline) James, Maria, and Tasha. Can you write that one down?
Teacher:	(to Tiara) So what you're trying to do is find the different ways the race could finish. With those three kids. Does that make sense?
Tiara:	Uh-huh.
Teacher:	(to Jacqueline) Jacqueline, do you see what I'm saying? You're trying to find all the different ways the race could finish. With those three kids. Like you wrote down two different ways now. Now you wanna try to find another way it could end. Do you guys understand?

In responding to Tiara's request, the teacher first asks Tiara about the kids who are participating in the race, makes explicit about the order of race, and elicits an example. After a quick conversation with the teacher, Tiara produces two possible results: (1) Tasha, Maria, and James and (2) James, Maria, and Tasha. In the previous task of the three-car train problem, students have difficulties with understanding how two solutions are different (e.g., grp and prg) as addressed by Riya. In working on the three-kids race problem, the students, including Tiara, have a better understanding that the arrangement of Tasha, Maria, and James is different from the arrangement of James, Maria, and Tasha.

The second type of request is to clarify one of conditions of the problem. During individual work, Marcellus asks whether he needs to use all of the three kids in a race.

Marcellus:	Do we have to use all three of these?
Teacher:	Do I have- do you have to do what?
Marcellus:	All three of these.
Teacher:	What do you mean all three?

Marcellus:	Like can we do two people finish?
Teacher:	No, because you wanna know the order of the first three places.
Marcellus:	Okay.
Teacher:	Yeah, that's a good question.

The problem states that someone finished first, someone finished second, and someone finished third, but the condition of "exactly only one" might not be very clear for students in the three-kids race problem, whereas "exactly only one" is explicitly stated in the three-train problem. The teacher clarifies that the problem asks for the order of the three places but does not address this issues to the whole class.

The third type of request relates to the allowance of repeating the name more than once. During individual work, Jacqueline asks:

Jacqueline:	Are you allowed to use a name more than once?
Teacher:	In the same race? Or in the- across the different ones you write down?
Jacqueline:	(nodding her head)
Teacher:	In the same race you can't use the same name twice, because somebody can't come in first and second. But that's not what you mean, right?
Jacqueline:	(nodding her head)
Teacher:	You mean what?
Jacqueline:	(silent)
Teacher:	Do you mean like in a different race? Like-
Jacqueline:	(nodding her head)
Teacher:	-yes, because it's the same three kids over and over. And you're trying to find different ways they could finish. Does that make sense?
Jacqueline:	(nodding her head)

Jacqueline's question could be interpreted in two different ways: (1) allowance of repeating the same kid in the same race; and (2) allowance of repeating the same kid across different races. The first issue relates to the conditions of the problem (i.e., exactly only once), whereas the second issue relates to the production of multiple solutions. By offering two interpretations, the teacher clarifies that it is not allowed to use the same kid twice in the same race but is allowed to use the same kid across different races. In some sense, the issue of repeating the same kid across different races is similar to Riya's concern for the three-car problem in the previous day but is mathematically different. This issue relates to what counts as "different mathematical solutions" for the there-car problem, whereas it relates to whether there are multiple trials of race for the three-kids race problem. If someone does not interpret the problem as multiple trials of racing of three kids, the criterion for ordering would not be a mathematical one. For example, Jana does not have mathematical comprehension of the problem and produces only one solution, while she produced five solutions for the three-train problem in the previous day.

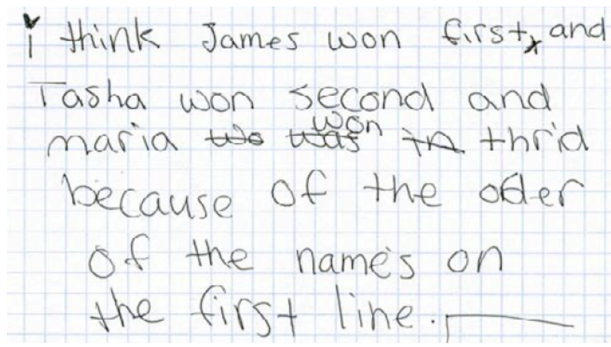


Figure 7.5. Jana's notebook writing for the three-kids race problem

After 17-minute individual work, the teacher convenes the class to launch a discussion about the three-kids race problem. The teacher asks Alvan to read aloud the problem, requests Levi to restate what the problem is asking, and then elicits an example from Marcellus.

Marcellus:	Uhm… sure. There's James, Tasha, Maria.
Teacher:	James- James, Tasha, Maria?
Marcellus:	Yeah.
Teacher:	(writing "James, Tasha, Maria" on the board) So is James the first place in this one? Is this like first, second and third like that?
Marcellus:	Yeah.

James
Tasha
Maria

Marcellus provides a solution of James, Tasha, and Maria and the teacher records it on the board while checking the order of racing. Unlike the three-train problem, the conditions of the problem are not probed further. The teacher continues to elicit another way of finishing the race.

Teacher:	Okay. Is there another way that the race could finish? Okay, Alvan.
Alvan:	Tasha, James, Ma-
Teacher:	Tasha…
Alvan:	James… Maria.
Teacher:	(writing "Tasha, James, Maria" on the board) Okay. Why is that one different? Can you explain how that's different from the first one, Alvan?
Alvan:	Uhmm…
Teacher:	How are these two races different? What's different?
Alvan:	Well now, the second one, Tasha is first and James is second. And Ma-Maria is third…
Teacher:	Okay, very good explanations. So, Tasha's first the second time, Jame-James is first the second time, but Maria is third both times. Okay?

James	Tasha
Tasha	James
Maria	Maria

After making a record of Alvan's solution on the board, the teacher asks for the difference between the first solution and the second solution instead of checking the conditions of the problem. The teacher translates Alvan's description about the order of finishing a race in his solution to the idea that the first place and the second place changes with each other, but the third place does not change.

Teacher:	Who'd like to give another race- another way that it could have finished? Yes, Aiyana?
Aiyana:	Maria, Tasha, and James.
Teacher:	(writing "Maria, Tasha, and James" on the board) Maria, Tasha, and James. So, what's different about that one from the other ones we've seen?
Aiyana:	They're the same people, but they're just in different orders.
Teacher:	Okay, well what's-what's the thing you really notice about this one compared to the first one? Can you say some-can you say one thing that's quite different in this one?
Aiyana:	Maria's first?
Teacher:	Maria's first. Maria didn't win in the other two. Right? So, that's different.

James	Tasha	Maria
Tasha	James	Tasha
Maria	Maria	James

After recording Aiyana's solution on the board, the teacher asks Aiyana how her solution is different from the previous two solutions. Aiyana provides an explanation in a broader term about different orders, but the teacher narrows the focus on the difference between the first solution and third solution (the first place and the third place change with each other, but the second place remains the same). Instead of noticing the places that have switched and the place that have not switched, Aiyana shares her observation that Maria won for her solution. In this process of translation, the teacher does not just repeat the words verbatim, but sketches possible structure of solutions for students building on what Aiyana said. The three solutions elicited have a different person winning the race.

Teacher:	Okay, does somebody have another one? Another race? Another example of how this could finish? Nina, do you have a different one?
Nina:	Uhm… Tasha, Maria, and James.
Teacher:	(writing "Tasha, Maria, and James" on the board) Tasha, Maria, and James? Okay, so which one is this most like, Nina? Which race is this one most like?
Nina:	Uhm, Tasha, James, and Maria
Teacher:	So it's most like this one (pointing out the second solution suggested by Alvan). And what's different about this one (pointing to the second solution suggested by Alvan) and this one (pointing to the fourth solution suggested by Nina)?
Nina:	Uhm, Maria is in second and James is in- James is in the second on the other one.
Teacher:	Okay, so here (in the fourth solution), Maria's second, here (in the second solution), James is second, but Tasha wins both of those. Nice, Nina.

James	Tasha	Maria	Tasha
Tasha	James	Tasha	Maria
Maria	Maria	James	James

After making a record of Nina's solution on the board, the teacher slightly alters the prompt. When the second solution and the third solution are proposed, the teacher asks how the new proposed solution is different from the first solution. After the three solutions that have a different kid in the first place, the teacher modifies the prompt into "Which race is this one most like?" Depending on the focal point, one might think that the fourth solution is the most like the second one (if focusing on the first place) or that the fourth solution is the most like the third one (if focusing on the third place). However, it is evident that the fourth solution looks least like the first solution because all places have different kids except that Tasha always finishes earlier than Maria. Focusing on the first place rather than the third place, Nina explains how the second place differs in both ways of finishing a race while making implicit that the second place and the third place are switched around in the second solution and the fourth solution. By adding that the first place is the same, the teacher pairs up the second solution with the fourth solution.

The class continues to collect solutions. Natania proposes her fifth solution of James, Maria, and Tasha.

Teacher:	Who else wants to tell us another one you came up with? Uhm, Natania.
Natania:	James, Maria, Tasha.
Teacher:	(writing "James, Maria, Tasha" on the board) James, Maria, Tasha. Which one is this one most like?
Natania:	James, Tasha, Maria.
Teacher:	Which one?
Natania:	James, Tasha, Maria.
Teacher:	(pointing to the first solution) The first one? James, Tasha, Maria. In both cases (the first solution and the fifth solution), James wins, but what's different between the two?
Natania:	Maria is second and last and Tasha is second and last.
Teacher:	Okay, so this one (pointing to the first solution), Tasha's second and this one (pointing to the fifth solution) Maria's second, but James wins in both of them.

James	Tasha	Maria	Tasha	James
Tasha	James	Tasha	Maria	Maria
Maria	Maria	James	James	Tasha

After recording Natania's proposal on the board, the teacher chooses a similar prompt as she did for the fourth solution. Instead of checking the conditions of the problem or finding difference with other proposals, the teacher asks Natania to choose the most like solution to hers. Similar to Nina, who chose the most like solution as having the same kid in the first place, Natania nominates that the first solution (James, Tasha, Maria) is the most like solution as the fifth one (James, Maria, and Tasha). Pointing out that the winner (James) is same in both solutions (the first solution and the fifth solution), the teacher asks about the difference between them to Natania. The class continues to collect another solution. Ricky proposes the sixth solution.

Teacher:	Is there any other one? Is there any other way that this could have finished? Ricky?
Ricky:	Maria, James, Tasha.
Teacher:	(writing "Maria, James, Tasha" on the board) Maria, James, Tasha. Which one is this most like, Ricky?
Ricky:	Maria, Tasha, James.
Teacher:	Maria, Tasha, James. What's the same about with this one (pointing to the sixth solution) with that one (the third solution)?
Ricky:	They're-Maria's still in first.
Teacher:	Maria's still on the first. And what's different?
Ricky:	James is in second (the Teacher is pointing at the sixth solution on the board) and Tasha (the Teacher is pointing at the third solution on the board) is in second.
Teacher:	Okay, James is in second here (pointing to the sixth solution) and Tasha's (pointing to the third solution) in second here.

James	Tasha	Maria	Tasha	James	Maria
Tasha	James	Tasha	Maria	Maria	James
Maria	Maria	James	James	Tasha	Tasha

After making a record of Ricky's solution on the board, the teacher asks Ricky to nominate the most like solution to his proposal. Similar to the previous two students, Nina and Natania, who interpret the most like solution as having the same kid in the first place, Ricky selects the third solution as the most like solution to his. Not even expressing the satisfaction with eliciting all the solutions, the teacher seeks another solution.

Teacher:	Another one? Elina? Another one?
Elina:	You can't do another one.
Teacher:	You can't do another one? Why not?
Elina:	You did all of them.
Teacher:	How do you know that?

Elina: Because we've- take the first one, for example, we've already switched Maria and- Tasha around, and the second one, Tasha, James, Maria-, we've switched Maria and James around and-with Maria, Tasha, and James (in the third solution), we've already switched Tasha and James.

Teacher: So, you're comparing this one (writing a blue arrow sign at the first solution) to this one (writing a blue arrow sign at the fifth solution)?

Elina: Uh-huh.

Teacher: So James wins in both of these, but then we switch Tasha and Maria around. Is that what you said?

Elina: Uh-huh.

Teacher: And then what's the next two you compared?

Elina: Tasha and James and Maria (the teacher is writing a green arrow sign at the second solution) and then Tasha, Maria, and James (the teacher is writing a green arrow sign at the fourth solution)

Teacher: And what's-what do we do there?

Elina: We just switched James and Maria.

Teacher: So we have Tasha first, but we switched James and Maria around?

Elina: Uh-huh.

Teacher: Okay, and what's the third one you compared?

Elina: Maria, Tasha, and James.

Teacher: And what's going on there?

Elina: Uhmm… we just switched around Tasha and James (the teacher is writing a red arrow sign at the third solution and at the sixth solution)

Teacher: So these are ones that are getting switched around (adding a red circle around James and Tasha for the sixth solution) in that one (adding a red circle around Tasha and James for the third solution), but Maria stays first (underlining Maria both for the third and sixth solution).

↓	↓	↓	↓	↓	↓
James	Tasha	Maria	Tasha	James	Maria
Tasha	James	Tasha	Maria	Maria	James
Maria	Maria	James	James	Tasha	Tasha

Upon the teacher's request for another solution, Elina asserts that no more answers can exist for the three-kids race problem. As a rationale for the exhaustiveness of the solutions, Elina uses the explanation provided by Nina, Natania, and Ricky in the process of pairing up solutions. Similarly, Elina fixes the kid in the first place and then switches the kid in the second place with the kid in the third place, thus pairing up the first solution with the fifth solution, the second solution with the fourth solution, and the third solution with the sixth solution. After Elina's explanation for three pairs, the teacher goes through each pair by marking the arrow with the same color and makes clear that each pair has the same kid in the first place but just switched the order for the kids in the second place and the kid in the third place.

Teacher:	Can someone say what Elina showed, about why she thinks that we have all of them? I tried to make some record of what she was saying, but I didn't quite finish doing it. Could somebody say how she's trying to explain that we have all the ways of making the race? Callie?
Callie:	Because all of them have won at least one or at least twice? Yeah.
Teacher:	Say that again? All of them what?
Callie:	All of them won at least two times or one.
Teacher:	Okay, all of them won at least two times. And then what?
Callie:	Or one.
Teacher:	Did anyone win just once?
Callie:	Uhm, no.
Teacher:	How many times did everybody win?
Callie:	Two.
Student:	Two.
Teacher:	Did they win exactly two or at most two or at least two?
Students:	Exactly.
Callie:	Exactly.
Teacher:	Exactly two. They each won exactly two times.

In teaching the three-kids race problem, some discourse moves such as asking to repeat , restate,or revoice are less loaded throughout the lesson. This is the first moment that the teacher calls for repeating the explanation. In scaling up the explanation provided by Elina who pairs up two solutions together which have the same kid in the first place, Callie explains that all of the three kids won the race at least once or twice. Through her exchanges with the teacher, Callie was able to choose the number of winning a race for each student (i.e., two) but still made implicit the term of "at least". To make the explanation accurate, the teacher asks whether it is "exactly" two, "at most" two, or "at least" two. After making explicit that each kid won exactly two times, the teacher reviews cases in which Tasha won but James and Maria switched the places and cases in which James won but Maria and Tasha switched the places. At this point, the teacher gives a space for individual students to reflect on whether he or she is convinced by Elina's argument that the class has found all of the possible ways to finish a race (see Table 7.1). After one-and-half minute partner talk, the class moves onto another mathematical task.

Table 7.1. The EML 2009 student's notebook writing about their convincement whether they have found all of the solutions

Convinced	Aiyana, Akilah, Amelia, Connor, Collin, Jana, Levi, Malik, Manley, Marlais, Marcellus, Mannis, Natania, Nina, Riya, Ricky, Tiara
Not convinced	Alvan, Jacqueline
No record	Dante, Elina, Evan, Tonya, Teri, Sandra

In the first session of Day 9, the three-train problem is re-introduced as a warm-up problem, while substituting a red rod with a yellow rod. After doing the daily routines, the students begin to work on the warm-up individually.

> Find all the ways to arrange the light green, purple, and yellow rods into three car trains using exactly one of each rod.
> How are you sure you have found all the ways?

After the 9-minute of individual work, the teacher convenes the class to collect solutions for the three-train car problem. The teacher asks Manley to read the first part of the problem and requests Elina to restate what the problem is asking.

Elina:	It's saying to rearrange- to take- to make a train with exactly one of each, and then- then keep rearranging it till you make the same.
Teacher:	What does exactly one mean? What is it saying to us about- when it says exactly one, what's that talking about? Riya?
Riya:	Only one- just that one.
Teacher:	Say that again?
Riya:	Only one.
Teacher:	Only one? So you can't do what? Tell me some things you can't do.

Elina's restatement is linked to clarifying what "exactly one" means in the problem statement. Through the process of elaboration by Riya, Tonya, Alvan, Jana, Levi, Aiyana, and Marlais, the term "exactly one" is developed into the general statement of "neither any numbers above one nor any numbers below one." Although Riya points out that "exactly one" refers to only one, the teacher consolidates the meaning building on the collective work done by students. The teacher does not use the term "condition" but effectively communicates that the answer should use exactly one of each rod.

The teacher asks Manley to read the second part of the problem. Afterwards, the teacher frames her question as how many different ways to rearrange the three rods. Compared to the previous two cases in Day 4 and Day 5 in which open the discussion by eliciting one of solutions, the teacher opens the discussion of Day 9 by eliciting the number of solutions for this problem. In addition, at the beginning of the discussion, the teacher clarifies the meaning of "exactly one," restrict to the three rods, and asks for rearranging. Marcellus asserts that there are six ways to arrange the three rods. Not satisfied with just the correct number of solutions proposed, the teacher checks whether there is anyone who has a different number of solutions than six and gets ready to make a transition to discussing a way to proving how they can be sure. At that moment, Callie suggests that she has seven solutions. One student sitting next to Callie reports that Callie uses the red rod, but the teacher provides an opportunity for Callie to reflect by herself. Changing the original question asking how they can sure that they have all the answers, the teacher elicits a solution one by one.

When each student proposes one of the arrangements, the teacher makes a record on the board and builds a train using Cuisenaire rods on the board. After three arrangements were proposed ("gyp" by Mannis, "gpy" by Natania, and "ygp" by Tiara), the teacher checks whether there are any duplicates and then makes sure that all of the proposed arrangements are different from each other. When three additional arrangements are proposed ("pgy" by Evan, "ypg" by Tonya, and "pyg" by Levi), the teacher makes a record on the board and builds a train using Cuisenaire rods on the board. In this process, no calls for repeating, restating, or revoicing are made by the teacher. Even after eliciting all different six solutions, the teacher continues to ask for another solution. Callie, who said that she had seven solutions at the beginning of the discussion, suggests ygp, but the class points out that her solution is already listed on the board. At this point, the teacher directs students' attention to their notebooks and then asks them to check whether anything is missing. Natania comments that Callie only found six solutions on her notebook, but the teacher does not problematize Callie's initial assertion that she had seven solutions. Alleviating Natania's concern about Callie's seven solutions, the teacher continues to collect another solution. Manley suggests ypg, but other students immediately point out that it is already listed on the board. After the duplicated proposals by Callie and Manley, the teacher checks again whether anyone has a seventh solution once again and proceeds to prove how to be sure that there are only six solutions.

Teacher:	Okay. So this is six right now, that's what Marcellus said. Does anyone have a seventh one? So how do we know that it's only six? It looks like people don't have a seventh one, but how do we know we're just not finding it? Is there anyway to be sure we're just not missing something? Jacqueline?
Jacqueline:	Uhm… We can be sure if you switched them all around and you get the same one again.
Teacher:	So you're saying if we switched them around and we get the same one, then it's not another one.
Jacqueline:	Yeah.
Teacher:	Is there any way to be sure that we've switched them around all the ways we can? Is there any way to be sure about that? Anyone have an idea about that? Is there any way to do that, or just- we just try until we don't repeat anything.

As a way to make sure that there are only six solutions for the three-train problem, Jacqueline tosses the idea of switching around until repeating the same solution but does not specify how it proves the exhaustiveness of solutions. The teacher further asks for all the ways of switching around and asks whether they can rely on the empirical evidence of repeating.

Marcellus:	Uhm… There are only six because you did two of each one, and you switched- so let's start with the G.
Teacher:	Can you come up and show what you're talking about?
Marcellus:	Sure.
Teacher:	Can people look at what Marcellus is doing, please? Don't be playing with your rods right now. But look at what Marcellus is showing, and see what you think about it.
Marcellus:	(comes to the board) So we have G, Y, P. We switched them around, and any more versions [...] we switch these around again (points to "gyp" and "gpy"). [...] this one. And then, the same

thing with these two (points to “ygp and “ypg”), and then these two (points to “pyg” and “pgy”).

Sandra: (whispering) I- I can't understand.

Teacher: Can someone repeat what they think Marcellus is saying? What do you think that Marcellus is saying? Aiyana?

Aiyana: I think he's saying that you know, because like- you keep like the same first letter, but you switch the other two letters around. Like Y was the second one, and P was the last one, and then for the next one, P was the second one and Y was the last one.

Teacher: Is that what you're saying?

Marcellus: Yes.

Teacher: Y can go in the first place then you switch the P and the G?

Marcellus: Yeah, and then you can't do any other for the Y, so the- with the Y being in front, so-

Teacher: So- so is this- so he's saying if you put- is this what you're saying, Aiyana? If you put Y first you could have then P and G (writes “ypg” on the board), and then you're saying if he says you did Y first you could switch those two (writes “ygp” on the board). And why can't you do another one, Marcellus?

Marcellus: Because if you did another one, you'd just be switching these (points to “gp” on the “ygp”) and that would equal this (points to “ypg”).

Teacher: Is that what you're saying?

Aiyana: Yeah.

Teacher: What do people think about that? Is that right? If the yellow can only go first two times? Tonya? That makes sense to you?

Adding onto Jacqueline’s idea of switching around, Marcellus provides an idea of pairing up two solutions together. Marcellus accepts the teacher’s invitation to the board and provides an explanation how the pair with the same first rod switches the second rod with the third rod. After Marcellus pairs up all of the solutions, the teacher asks Aiyana to repeat Marcellus’ explanation. Beyond pairing up two solutions, Aiyana explains that Marcellus keeps the first letter the same but switches the last two letters. The teacher asks for an explanation for another pair and Marlais provides an explanation for gpy and gyp. The teacher asks for the explanation for another pair and Jana provides an explanation for pgy and pyg.

Teacher: Can someone explain how Jana used that same idea? These are the two that Jana gave us. You can sit down, Marcellus. Thank you. Can someone explain- can you see this? P G Y, P Y G. Can someone explain how Jana used the idea that Marcellus and Aiyana were talking about, and that Marlais used? Manley?

Manley: She just switched the G and the Y around.
Teacher: She just switched the G and Y? Is that what you did?
Jana: Uh-huh.
Teacher: Okay, so one, two, three, four, five, six. Does that convince you that there can't be more than six? Is there anybody who's not sure about that? What other problems have we done that are like this? Malik, you're not sure? Okay. Let's stop for a minute. Sorry, I didn't even stop. So Malik's not sure that shows there are only six. Can you tell us why you're not sure?

After the explanation for the pair of pgy and pyg, the teacher checks whether the method convinces them that there are only six solutions. In checking whether anyone is unsure about the method, one of students, Malik, addresses that he could not fully understand what the problem is asking. Not even ignoring one student's struggle, the teacher returns to the problem statement and then asks Nina to explain what the problem is asking. Following up on Nina's restatement of the problem, the teacher checks whether Malik understands what the problem is asking and whether the six solutions make sense to him. Throughout this process, the structure of six solutions are not just discovered by one individual student, but are built collectively.

- Jacqueline suggests the idea of switching around until it will be repeated
- Marcellus pairs up two solutions which are switched around for all cases (gyp and gpy; ygp and ypg; pyg and pgy)
- Aiyana repeats Marcellus' explanation but she makes explicit that Marcellus' method keeps the first letter the same but switches around the second letter and the third letter with the illustration of how y and p are switched around in the second and third.
- Marcellus confirms Aiyana's explanation
- The teacher adds that y goes in the first place in Aiyana's illustration
- Marcellus asserts that there are no more cases in which y is in front beyond the two solutions because switching gp in ygp again produces ypg
- Aiyana confirms Marcellus' explanation
- Marlais provides an explanation for gpy and gyp
- Jana provides an explanation for pgy and pyg, Manley repeats Jana's explanation and Jana confirms Manley's repetition.

The teacher makes a transition for generalization by asking whether any other problems they have been working on are similar to the three-train problem with purple, yellow, and light green. Marlais nominates the three-kids race problem. After Marlais reads the problem aloud, the teacher asks him about the results of the problem.

Teacher:	So what did we find out? Can you show us the results?
Marlais:	We found out, like- what Marcellus said, we use the names twice, and then we switch these two around to make a different chart right there.

In explaining the finding of the three-kids race problem, Marlais makes a reference to Marcellus' initial explanation about the three-train problem. It is implicit that each name is appeared in the same place twice, but the explanation provides enough details about the basic structure of the solutions. After Marlais' explanation, the teacher seeks for comments, agree, disagree, or questions.

Teacher:	Comments about what Marlais said? Remember when I ask about comments, you can say you agree, you can ask a question, you can say you disagree. What do you think about what Marlais is saying? You said you think that problem is very similar to the one we did today. So, now I'd like a comment on that. What do you think about what Marlais is saying? I'm seeing a lot of the same hands. How about some different people? What do other people think about what Marlais just said? Elina?
Elina:	I think- well, I agree, because in the warm up problem, we switched around- for example, in Y P G and Y G P, we just switched around the P and G. And in this we switched, for example, we- for example with James, Tasha, Maria, and then James- James, Maria, and Tasha, you... switch around Tasha and Maria.
Teacher:	Okay, so you're saying- you're showing how they're the same. You put something first-
Elina:	Yeah.
Teacher:	And then you switched the second two? Is that what you were saying, Marlais?
Marlais:	Yes.
Teacher:	Okay, other comments? Alvan?
Alvan:	I agree.
Teacher:	You agree with that? Why do you agree?
Alvan:	I'm only talking about what- like the similar things?
Teacher:	Sorry?
Alvan:	Are you talking about like the similar-

Teacher:	Yeah. The thing- those are the same- why are they the same?
Alvan:	Like, because it like- because it has like the two like, like all two people. Like James and James, and Y- and like Y P G. And Y P G. And Tasha- and two Tasha's. But when it's cars in the first, like G Y- G P Y and G Y P. And- and- and there's two G Y P's and then now, now Maria came in first- came in first two times. And like G- like... G P- P G Y...
Teacher:	So I think what- I think what you're saying, Alvan, is that like- Maria can come in twice first. Like Y can come in first place twice, and then we switch around- whoever. James and Maria. James and Maria, and then James can come in first and we switch around Tasha and Maria. Are you seeing that to be the same?
Alvan:	Yeah.
Teacher:	Is that what you were saying?
Marlais:	Yeah.
Teacher:	Okay, very nice explanations from everybody. Does anyone else wanna comment? Jacqueline?
Jacqueline:	I was-
Teacher:	You can sit down, Marlais. Thank you.
Jacqueline:	I agree because Elina was like- I agree that like- because it's the same with the rods. All you do is like switch and switch and switch.
Teacher:	Uh-huh. Is that what you were saying, Alvan?
Alvan:	Huh?
Teacher:	That's what you were saying too? That- like it's the rods, you're just switching the back pieces.
Alvan:	Yeah.

Three students, Elina, Alvan, and Jacqueline, comments on Marlais' explanation. In positioning an agreement, Elina provides a specific illustration of Marlais' explanation. By mapping y to James, p to Tasha, and g to Maria, Elina explains how the switch-around method works in the pair of two solutions for each problem (ypg and ygp; James, Tasha, Maria and James, Maria, Tasha). Following up on the specific example provided by Elina, the teacher synthesizes it using a more general statement (i.e., put something first and then switching the last two around) and then confirms it with Marlais. In expressing an agreement, Alvan expands the method for a single pair (two cases in which James in the first place) to all three pairs (two cases in which James in the first place; two cases in which Tasha in the first place; two cases in which Maria is in the first place). After hearing Alvan's explanation, the teacher summarizes his explanation again and then confirms again with him and Marlais. In expressing an agreement, Jacqueline makes a general statement of switching pieces around. The teacher again checks with Alvan whether it is what he meant.

- Marlais provides an initial explanation.
- Elina illustrates a single pair and Marlais confirms.
- Alvan illustrates all of the three pairs and Marlais confirms.
- Jacqueline provides a general statement and Alvan confirms.

In summarizing the explanation, the teacher makes the structure of solutions more salient to students.

Teacher:	So everybody gave different kinds of good explanations. One thing that Alvan did that was especially important- I just want people to notice. Can everyone look up? So in Alvan's explanation, he was very explicit. I think Marlais was, too. But I'm just- both of you saying like this is like Maria being first twice.
Teacher:	The Y is like Maria, and this is like Tasha and James. Tasha and James, James, Tasha. So they really connected the two problems to show us how they're the same. How many different races were there for that problem? How many dis- different race orders were there? Nina?
Nina:	Six.
Teacher:	Six, and how many different arrangements here?
Nina:	Six.
Teacher:	What if I gave you three different colors or rods? What if I said purple, orange, white. How many arrangements could you get for that? Jac- Natania?
Natania:	Six.
Teacher:	What if I said gray, dark green, blue? How many arrangements could you get for that? Jana?
Jana:	Six.
Teacher:	Six. Does it matter what I- what things I put in it? So can someone say, in general, what we figured out about rearranging three things? What do we know about rearranging three things now? If you say it doesn't matter what color or races or whatever, can someone say what the general thing is we know? If you have three things and you wanna put them in different arrangements, what do we know about that? You wanna say it, Jacqueline?
Jacqueline:	That they will always equal six.

In doing this, the teacher makes visible the core idea of the problem (i.e., rearrangement of three things in which the order of arrangement matters) at the forefront, while making the unimportant contextual features in the background (i.e., the color of rods, whether it is a train-building or race). The teacher asks students to write what they found in the notebook and makes a transition into another mathematical task.

7.3. The Case of EML 2010

Preview

In the EML 2010, the three-digit problem is introduced as a warm-up problem of Day 7 and the three-car train problem is introduced as a warm-up problem of Day 9. Like other warm-up problems, each student starts to work on the problem without any formal set-up. On Day 7, the whole-group discussion begins by clarifying the terms (i.e., three-digit; only once) and then elicits the conditions of the problem: (1) 3-digit number; (2) use the digits 1, 2, 3; and (3) use each digit only once. Instead of taking up one individual student's complete list, the teacher elicits solutions one by one: 321 by Kassandra, 321 by Ella, 132 by Ahmed, 231 by Hala, 213 by Macaulay, and 123 by Javonte. After eliciting all of six solutions, the teacher does not express that the class has found all of the solutions, but continuously seeks other solutions. After the class eliminates the duplicated proposals, Jaclyn claims that there are no more answers. Devante explains the reason as repeating and Thailee explains the reason as putting them in order. Using Jaclyn's method, the class reorganizes the proposed solutions in an order: 123 and 132 by Thailee; 213 and 231 by Shelly; and 312 and 321 by Eric. The class moves onto the similar problem of arranging 3, 5, and 7 and lists six solutions by using Thailee's method: 357 and 375 by Bernard; 537 and 573 by Devante; and 735 and 753 by Samara. On Day 9, the whole-group discussion begins by sharing the number of solutions produced. After Ella proposes six solutions, Thailee makes a generalization that arranging three things produces six solutions. After ypg is proposed by Ahmed, the next list is proposed by using a structure: ygp by Eric, gpy by Jaclyn, gyp by Madeline;, pgy by Jason, and pyg by Michael.

Extensive Detailed Analysis

The three-digit problem is introduced in the first session of Day 7, assigned as a warm-up problem. Like other warm-up problems, each student starts to work on the warm-up problem of Day 7 after doing some daily routines without any formal set-up of the problem in a whole-group setting.

How many different three-digit numbers can you make using the digits 1, 2, and 3, and using each digit only once? Show all the three-digit numbers that you found. How do you know that you found them all?

After Javonte reads aloud the problem, the teacher asks Thailee to explain what the problem is asking.

Thailee:	It's asking you to put the one, one -put the digits one, two, and three together and make as many three digit numbers as you can.
Teacher:	Okay, what is a digit? What's a digit? Everybody did the problem, so I know you've figured out what it means, but what is it? Jaclyn?
Jaclyn:	It's three rows of three numbers.
Teacher:	Well that's a three digit number, but what does just the word digit mean?
Jaclyn:	Number?
Teacher:	Like you knew when I said three digit number what kind of a number I was asking you to make, right? What?
Jaclyn:	(silent)
Teacher:	Can someone give an example of a number that's not a three-digit number? Shelly?
Shelly:	Five.
Teacher:	Okay, why is five not a three digit number? (writes "5" on the board)
Shelly:	Because there's only one number and not three of them.
Teacher:	Is there a digit there?
Shelly:	Uh-huh.
Teacher:	What digit?
Shelly:	Five.
Teacher:	Five. So what kind of digit number is that?
Shelly:	A one digit number.
Teacher:	A one digit number.

Thailee's restatement is fairly sufficient in capturing the original statement of the question, but misses one important condition of the problem—using each digit only once. Instead of being fastidious about capturing all of the conditions of the problem in the first trial of restating the problem, the teacher uses Thailee's restatement to attend to the vocabulary of digit and three-digit number. The teacher attempts to define the term digit, but Jaclyn defines the term three-digit number. When the teacher's reattempt to define the term digit expands to a broader mathematical concept of "number," the teacher elicits a non-example of three-digit number. Shelly proposes 5 as a non-example of three-digit number, which is further used to make a distinction between a digit (5) and the digit-number (1).

Teacher:	Can someone give a two-digit number? Macaulay?
Macaulay:	Two.
Teacher:	Okay, why is this a two-digit number?
Macaulay:	Because it only goes up to two.
Teacher:	Okay, is that what digit means?
Students:	No.
Teacher:	No, this is the number two, but what would be a two-digit number Macaulay?
Macaulay:	A two-digit number would be you can't go no higher than two.
Teacher:	No that's not what it means. A digit is a number, so I'm looking for a number that has two digits in it. Can you tell us a number that has two digits in it?
Macaulay:	One two?
Teacher:	Okay, what is that number? (writes "2" on the board)
Macaulay:	Twelve.
Teacher:	Twelve. Is twelve a two-digit number?

To consolidate students' understanding of three-digit number, the teacher continues to elicit a non-example of three-digit number. At this time, the teacher asks for an example of a two-digit number. In proposing 2 as a two-digit number, Macaulay misunderstood a two-digit number as digit 2. Faced Macaulay's misunderstanding about the key term, the teacher immediately corrects his misunderstanding and clarifies the meaning of the two-digit number. Macaulay revises his initial proposal of 2 to 12. As another non-example of three-digit number, the teacher asks for an example of four-digit number.

Teacher:	Could somebody give a four-digit number? Michael?
Michael:	Uhm… a thousand?
Teacher:	How do I write a thousand?
Michael:	One, zero, zero, zero.
Teacher:	Okay, does that have four digits? (writing "1000" on the board)
Students:	Yes.
Teacher:	What digits are in the number one thousand? Elias?
Elias:	One and three zeros.
Teacher:	One and three zeros, good.

Michael proposes 1000 as a four-digit number and the teacher checks whether there are four digits in 1000. After the teacher elicits non-examples of three-digit number (i.e., 5 as one-digit number, 12 as two-digit number, and 1000 as four-digit number), the teacher starts to elicit an example of three-digit number.

Teacher:	Okay so we're making three digit numbers, can someone give me one of the three digit numbers that you made? Kassandra?
Kassandra:	Three hundred twelve.
Teacher:	Three hundred what?
Kassandra:	Twelve.
Teacher:	Okay. (writing "312" on the board) Kassandra, can you show that that is one of the numbers by checking the conditions? What did the problem ask you to do?
Kassandra:	Asked me to find a three digit number.
Teacher:	Is this a three digit number?
Kassandra:	(nodding her head)
Teacher:	Okay that's the first thing, what else does it say?
Kassandra:	That I could only use one, two, and three one time.
Teacher:	Did you use only the digits one, two, and three?
Student:	Yes.
Teacher:	And what does the last thing say? Using?
Kassandra:	Each digit only once.
Teacher:	Did you use each digit only once?
Kassandra:	(nodding her head)
Teacher:	Okay good job.

The first solution (312) is proposed by Kassandra. After Kassandra's proposal, the teacher checks with Kassandra whether her proposal meets the conditions of the problem: (1) three-digit number; (2) only use the digits 1, 2, and 3; and (3) use each digit only once. Without further asking for repeating, revoicing, agreeing, disagreeing, or commenting, the teacher moves on to elicit another solution to the problem.

Teacher:	Who can give another three-digit number and explain why you know it's one of the ones for this list? Ella?
Ella:	Three hundred and twenty one.
Teacher:	Okay can you show why that's one of the answers to this problem, can you go back to this (pointing to the warm-up problem written on the poster)?
Ella:	It's one of the answers because I'm – it has three digits and I'm using only -
Teacher:	It has three digits.
Ella:	- the numbers one, two, and three. And I'm using each digit only once.

Similarly, after the second proposal (321) is made by Ella, the teacher checks how it meets the conditions of the problem. After eliciting two examples, the teacher makes explicit about the conditions of the problem and numbers them in the poster. Anthony identifies that it has to be a three-digit number and Bernard identifies that it has to use the digits one, two, and three and it has to use each digit only once. The teacher clarifies the meaning of "only once."

Teacher:	Use digit only once. What does it mean to use each digit only once? Can someone explain what that means? We've had all kinds of words like "no more than", "no less than", "exactly". What does only once mean? Zahara?
Zahara:	No more than one-no more than one time.
Teacher:	No more than one time. Can you use it zero times?
Zahara:	No.
Teacher:	Why can you not use it zero times?
Zahara:	Wait … Uhm…
Teacher:	So what I'm asking is could this be one of the numbers? (writing "32" on the board) That doesn't use the number one, is that okay?
Zahara:	No.
Teacher:	Why not?
Zahara:	Because it has to be a three digit number.
Teacher:	Good, so the rest of the problem tells you that you would have to use them every time. Okay?

Through the short conversation with Zahara, the teacher clarifies that it cannot use the number zero times and checks it using the example of 32. Using zero times not only violates the condition of exactly once, but also violates the condition of three-digit number.

In the similar way, Ahmed explains why 132 is an answer for the problem using the three conditions of the problem. Hala proposes 312 but other students respond that they already have that solution. With the teacher's response for another solution which is not listed on the board, Hala proposes 231 and explains why it could be accepted as a solution using the three conditions of the problem. Macaulay proposes 213 and explains using the three conditions of the problem. Javonte proposes 123 and explains why it could be accepted as a solution using the three conditions of the problem. On the board, the six solutions are listed.

312
321
132
231
213
123

Even after eliciting all of the six solutions for the problem, the teacher does not express that the class has found all of the solutions, but continues to elicit another solution. Coretta suggests 132 but other students respond that they already have it on the list. Jaclyn asserts that there are no more answers, but the teacher checks whether anybody has another solution for the problem. As no one proposes further solution, the teacher moves onto proving the exhaustiveness of solutions and makes a space for students to talk with their partners. After one-minute partner work, the teacher convenes the class to open a whole-group discussion for proving.

Devante:	Because it's like, what I told Eric is that uh, that uh, if like you try to make some, some other ones and then when you make that number you would look back up on the ones that you had before and then you'll see that you had the same number.
Teacher:	Okay, could someone say what Devante just said? Who understood what Devante just said? That was a very complete, you said that very completely. Who was listening to what Devante said and can say what he said? Boy I should see everybody's hand up when I ask a question like that. Did you not hear Devante? Okay I would like you to say that again nice and loud and I want everyone listening to see if you understand what he's saying. Go ahead.
Devante:	What I told Eric is that when you try to make some other numbers and then when you make that number you would look back up on your numbers that you made before and you'll see that you have the same number.
Teacher:	Okay who can repeat what Devante just said? Who understood what he was saying and can explain what he just said? Elias?
Elias:	He said that you can only use a number once because if you try to make another number you're going to have the same number.
Teacher:	Okay, so what is he saying? What's his reason that we can't make more? Coretta?
Coretta:	Because you're going to start reproducing the same numbers over and over and over and over again.
Teacher:	(To Devante) Is that the same thing you're saying, if she says you start reproducing the same numbers?
Devante:	(nodding his head)

Teacher:	Okay, so Devante's argument is if you start to make another-try to make any more you're going to repeat yourself.

Devante explains that he repeats solutions after producing six solutions and the teacher asks other students to repeat Devante's explanation. As the teacher does not see a lot of hands to volunteer to repeat Devante's explanation, she asks Devante to repeat his initial explanation. After Elias and Coretta explain Devante's reasoning, the teacher receives a confirmation from Devante whether Elias and Coretta capture his reasoning well. It might be subtle, but Devante's reasoning is elaborated through repeating and restating. More specifically,

- Devante expresses that if you make another number, you will see that you have the same number on the list.
- Elias adds that repeating the same solution is not counted based on Devante's reasoning
- Coretta elaborates the language as reproducing the same number over and over.
- The teacher elaborates the language as repeating oneself.

At this point, not pursing further Devante's reasoning of repeated solutions, the teacher makes a transition to another reasoning.

Teacher:	Does anyone have another way that they think we could explain that there's no way to find a seventh number? How could you be sure that there couldn't be another number. So Devante has one way, you could try making them using the conditions. I assume you meant using the conditions. Did you mean using the conditions? And you'd repeat yourself. Does anyone have any other ideas of how we could be sure? Thailee?
Thailee:	You could make it in order.
Teacher:	Could you speak up a little bit please?
Thailee:	You could make it in order, like one, two, three the regular numbers just all together -
Teacher:	Okay so one, two, three and then what? (writing "123" on the board)
Thailee:	And then you would do all the other ones and that means that it's only one more one.
Teacher:	So what do you mean by the other ones, what would you make?
Thailee:	The one three two.

Teacher:	So you’re saying you make one, two, three, and then you make one, three, two. (writes “132” on the board) What are you doing when you make those two?
Thailee:	You’re putting them in order. You do one then the twos then the threes.
Teacher:	Okay. And then this one? (pointing to “132” on the board)
Thailee:	Two, one, three.
Teacher:	Okay so what were you saying?
Thailee:	Oh.
Teacher:	Say it again.
Thailee:	That you’re putting them in order. So then you can make sure that you have them all.

Thailee suggests the idea of reorganizing the solutions in an order. When the teacher asks Thailee about the reasoning behind listing 123 and 132, Thailee names the method as ordering and implies that she reorganizes the solutions by putting one in the hundreds place, putting two in the hundreds place, and then putting three in the hundreds place. Thailee moves to the third solution (213) according to her recording system, but the teacher pays attention to the method of ordering the solutions.

Teacher:	Okay does anybody understand what Thailee is trying to show? What would be the next step in what Thailee is saying if we kept going with the way Thailee is working? She started the first two numbers, what would you do next to use the way Thailee is trying to show? What would you do next? Shelly, what would you do next?
Shelly:	We would start with the two hundreds and uh-
Teacher:	Okay.
Shelly:	-go two, three, one-
Teacher:	(writing “231” on the board)
Shelly:	-and then two, one three.
Teacher:	(writing “213” on the board) Thailee, is this what you would do next in the way you were thinking about it?
Thailee:	You do two, one, three first.
Teacher:	You would put the three first, but if she chooses to do the two first is that the same method?
Thailee:	Huh?
Teacher:	She’s putting the two first -
Thailee:	No, I mean in the numbers so they can be in order.
Teacher:	Okay, so let’s make a small change, because she has a – can I just let her change the one thing that was different? Do you mind, Shelly? What would you do? I know what you’re saying.
Thailee:	You would change the middle numbers.

Teacher:	You would want this to be two, one, three, first right?
Thailee:	Yeah.
Teacher:	Okay. And why do you want that? (changing the order of two numbers by writing "213" first and "231" next)
Thailee:	So then you could go in order and say-
Teacher:	-Okay-
Thailee:	-well I already have two thirteen.
Teacher:	You want them to go in order, so what would the second row be then, after two one three? Shelly, can you complete it, if she puts two one three first what goes next?
Shelly:	Two thirty one.
Teacher:	Two thirty one. So, Shelly, can you explain how the method works? What did you do when you put these two on?
Shelly:	Well you can – what I did on my notebook is I split them up into rows or columns and I had ones twos and threes and I first did ones and I went – I did one two three and one three two and um and then I went on to the twos and did two three one and two one three [and three two one and three one two.]
Teacher:	[Okay Shelly, if we] had the projector set up right now I would ask you to come show your notebook because I think it would be really nice to see it, but I'm not going to turn it on right this minute. Maybe you could show that another time. Because it sounds like you had a nice system in your notebook, but the projector is not set up right now, is that okay? Could you take what you were saying and explain what Thailee was doing and what you did when Thailee's argument right here? What did you do next? Thailee did the ones with the one hundreds first and what did you do?
Shelly:	Uhm… I did the two hundreds, but I didn't do it – I just did it in order by the first row, but not the second row because -
Teacher:	But, okay
Shelly:	I didn't really thought that mattered.
Teacher:	Okay, so Thailee, are you saying there wouldn't be another one with two hundred first?
Thailee:	(shaking her head)
Teacher:	Why not? How do you know that there are only two ways to put the two hundred first?
Thailee:	Because you already have a two in the front and you already have the other numbers.
Teacher:	What are the other numbers after the two?
Thailee:	One and three.
Teacher:	One and three and then?
Thailee:	Three and one.

After checking whether other students understand Thailee's method, the teacher gives other students a turn to apply Thailee's reasoning. Shelly starts with 2 in the hundreds place, but lists 231 first and then nominates 213 later. If only paying attention to the digit in the hundreds place, Shelly's method might be considered as ordering but not as rigorous as Thailee's method. The teacher checks with Thailee whether Shelly's method reflects the same method as Thailee's method. Shelly's method does not get an approval from Thailee, so the solutions are reorganized (213 and then 231). When the teacher asks why the method works, Thailee explains that she puts them in order. After changing the order of 231 and 213, the teacher checks with Shelly again why the method works. Although Shelly insists on keeping her original order of solutions (231 and then 213) and thinks that it does not matter to put them in the order in the tens place, she acknowledges that her method puts numbers in the order in the hundreds place but not in the tens place. At this point, the teacher asks Thailee about the exhaustiveness of two solutions that begins with 2 in the hundreds place. The teacher gives an opportunity to other students to apply Thailee's reasoning.

Teacher:	Okay, who can make the third part of Thailee's argument? Shelly made the second part. Thailee started with this, she put the one hundred first and switched around the two and the three, then she put the two hundred first, Shelly did it, and switched around the three and the one. What would be the third part of the argument? Does anyone see where this is going? How would you finish this? Eric?
Eric:	Um, put three hundred twelve.
Teacher:	Put the three hundred first? So can you tell us what the numbers would be?
Eric:	Twelve.
Teacher:	Okay.
Eric:	Three hundred and twelve.
Teacher:	Three hundred and twelve and then what?
Eric:	Then three hundred and twenty one.
Teacher:	Can you explain that, Eric?
Eric:	Um… because you're going in order so the first -
Teacher:	Speak up please. Speak louder.
Eric:	If you're going in order the first number is going to be three hundred and twelve-
Teacher:	And then that?
Eric:	-and the next number is going to be three twenty one.
Teacher:	Okay. So, Thailee, now can you – this is your argument right?
Thailee:	Uh-huh.
Teacher:	And you helped to build it.

The initial list of solutions	The reorganized list of solutions
312 (proposed by Kassandra) 321 (proposed by Ella) 132 (proposed by Ahmed) 231 (proposed by Hala) 213 (proposed by Macaulay) 123 (proposed by Javonte)	123 and then 132 (proposed by Thailee) 213 and then 231 (proposed by Shelly) 312 and then 321 (proposed by Eric)

By applying Thailee's method, Eric lists 312 first and then 321 later. After Eric's explanation, the teacher checks with Thailee whether Eric follows Thailee's reasoning. After reorganizing six solutions by applying Thailee's reasoning, the teacher gives Thailee a turn to explain why a seventh solution does not exist. Thailee comes to the board and explains.

Thailee: You will put them in order one through three and then you would put the lowest one first (points to "123" on the board) and then you put the highest one (points to "132" on the board) so you, you put the highest one next and then you would know that there is not a next one that comes higher than thirty two.

Teacher: Then what do you do next after you put the one hundreds?

Thailee: You put the two hundreds and then you put the lowest one first (points to "213" on the board) and the highest one (points to "231" on the board) and then you know that there is not another one that you can do because you can't use two over again.

Teacher: Okay.

Thailee: And then three hundreds the lowest first (points to "312" on the board), the highest (points to "321" on the board) and then you know there won't be any more because you can't go any higher.

Teacher: Who thinks they can explain Thailee's argument? When I say argument, what I mean is she has a way of showing that we can't make any more. It's kind of like a proof, or proving something. Can someone explain how she's proving that you couldn't make any more numbers?

After Thailee's explanation, the teacher asks whether someone could explain Thailee's argument but only Karina raises her hand. By asking students to write whether they are convinced by Thailee's argument, the teacher creates a private space for students to reflect on what they have discussed. As the teacher asks how many students are convinced by Thailee's argument, all of the students raise their hands. Instead of asking to re-explain Thailee's argument, the teacher now launches a new problem. She asks students to use Thailee's method to make all the possible three-digit numbers using 3, 5, and 7. The teacher changes the peripheral feature of mathematical problem but preserves the mathematical features. After a short individual work about one-and-half minute, the teacher starts to collect solutions for the three-digit number using 3, 5, and 7. After Bernard proposes his first pair of solutions of 357 and 375, the teacher asks Ahmed how it is similar to Thailee's method.

Teacher:	Okay, how is that the same as what Thailee did? Ahmed?
Ahmed:	It's not the same numbers but like we're doing the same thing by putting them in order lowest to highest.
Teacher:	Okay you're putting them in order lowest to highest starting with three hundred and then what's the next thing you're doing after that, Eric? After you put down the three hundred, what did you do?
Eric:	You put the next lowest number.
Teacher:	Which were what?
Eric:	Three seventy five.
Teacher:	So she did three five seven and then
Eric:	Three seventy five.

Beyond the superficial feature of the numbers that are used to arrange, Ahmed is able to see the structure of the solutions, putting them in numerical order from the lowest to the highest. According to this logic, after fixing the number in the hundreds place, Eric is able to order numbers from the lowest to the highest.

Devante proposes the second pair of solutions (537 and 573) and Samara proposes the last pair of solutions (735 and 753). After listing all of the six solutions in the order, the teacher checks with Thailee whether this is the same method that Thailee used for the first problem of making three-digit numbers using 1, 2, and 3. The teacher provides private space for individual students to write in their notebooks whether they are convinced that they have all of the solutions. After one-and-half minute individual notebook writing, each student shares what he or she wrote in the notebooks with a partner for a minute. After that, the teacher makes a transition to another mathematical task.

On the first session of Day 9, the three-car train problem is introduced as a warm-up problem.

Find all the ways to make different trains using exactly one each of the red, light green, and purple rods. Keep track of each train. How many are there? How do you know you have all the possible train using just those three rods?

Before launching a whole-group discussion, the teacher shares her observation that most students have the same number of solutions but use different methods of recording and different explanations. In opening up the discussion, the teacher first asks about the number of solutions for the problem. Ella asserts that there are six trains and nobody suggests different number of trains. The teacher gives Thailee a turn to comment about the six trains.

Thailee:	Uhm, I, I know that that would be six, because every time I do it, three-digit uhm, like three-digit number or letters and then it always ends up being six that I can do.
Teacher:	Can someone say what Thailee just said please? Who heard what Thailee said? Can someone just repeat to the class? Cause she said something about- I want to hear from you. Zahara, what did she say?
Zahara:	She says that when, whenever see that like three-digit number of something that, make up of three or something that she always ends up getting the answer six.
Teacher:	Okay, can someone else say what Zahara just says and I am gonna write down what Thailee's idea was? What did Thailee say about whenever we have three things?
Students:	(silent)
Teacher:	Is Zahara the only person who heard it? I had one person repeated. Thailee said something and only one person repeated. Who else besides Ahmed, Jaclyn, and Zahara heard it?
Students:	(few students raise their hands)
Teacher:	Okay. Then I am gonna ask Thailee one more time as clearly as possible, so everyone can listen to closely to what she is saying. Cause it's an important idea, I wanna make sure if everyone hears, then we can decide if we think true. So everyone look at, please look at Thailee and listen carefully.
Teacher:	(to Thailee) And you are trying to say that again one more time. I'm gonna write while you are talking, so.
Thailee:	Uhm, I, every time I do three-digit something then it equals up to six. Uhm, six all together how many I can do.
Teacher:	[writes what Thailee said on the board] Can somebody say? I'm still writing, so help me hear for somebody repeating what she said. Ahmed, are you thinking you can?

Ahmed:	She was saying that every time, she does like three-digit something, it always ends up to being six.

Thailee makes a connection between similar mathematical tasks that the class has been working on for the last few days and makes a generalization. Even though Thailee misses some important mathematical aspect in her explanation (ordering matters; using only once), she is able to see the same mathematical structure across several mathematical tasks. After Thailee's comment, the teacher asks Zahara to repeat what Thailee commented. The teacher asks other students to repeat Zahara's explanation but does not see enough hands raised. Because of the importance of hearing Thailee's comment, the teacher asks Thailee to repeat her initial comment and then gives Ahmed a turn to repeat Thailee's comment. The teacher further asks Ahmed to clarify the ideas of three things and six things, but Ahmed is not able to elaborate on the language. After checking with Thailee about her comment again, the teacher names Thailee's comment as a conjecture and continues to make an effort to elaborate the idea. To do that, the teacher reminds students about the similar problems (1-2-3 problem; a-b-c problem; purple, yellow, green problem) and then asks what they do for these problems.

Thailee:	I wanna say that maybe clear that more. Maybe when you do something like, if there was how many two, two-digit, you could do maybe end up with being four all together. Like you will-
Teacher:	So you are thinking now was two things instead of three, what the pattern would be, right?
Thailee:	(nodding her head)
Teacher:	Let's stick with three for a minute. That's very good idea to think about, but let's see if this said more clearly first.

In responding to the teacher' request to elaborate on the language, Thailee extracts a pattern from arranging three things and then applies it to arrange two things. Thailee expresses the number of solutions for arranging three things as the algebraic equation of 3×2. In this algebraic equation, 3×2 is a special case of n equals to 3 in $n \times (n-1)$. However, Thailee's reasoning is based on $n\times2$, instead of $n \times (n-1)$, which leads to the $2\times2=4$ for arranging two things. Although her interpretation of 2 in 3×2—interpreting a constant (2) rather than another variable (n-1)—is not mathematically correct, her attempt to transfer the number of solutions to construct a pattern is very important mathematical skill. Faced with Thailee's attempt to make a generalization—in spite of incorrect one—, the teacher does not diverge into another mathematical issue, but keeps the focus on elaborating on the language in Thailee's conjecture. The class elaborates on the language into "Every time we arrange three things into groups or into an order, we end up with six, you can pick answers, combinations, or solutions." Each student writes the statement in the notebook.

The teacher convenes the class to explore whether Thailee's conjecture is true. Instead of extracting all of the solutions from one student, the teacher elicits one solution from each student. Ahmed proposes the first solution of ypg. Unlike Day 7, the class does not discuss whether Ahmed's proposal meets the conditions of the problem. After his proposal, the teacher expresses her expectation to list solutions in a way that reveals the structure of solutions. The class had an extended discussion about the structure of solutions for the three-digit number using 1, 2, and 3, as well as applying the method for making three-digit number using 3, 5, and 7 on Day 7. Unlike the previous lessons on Day 7, which listed solutions as proposed first and then reorganized later, the teacher shows a clear expectation to propose solutions following a specific structure. The structure is more visible for listing the three-digit numbers, ordering from the smallest to the largest one, but listing the solutions in order does not apply for the three-train car problem. Instead, students need to be able to see the more fundamental structure, fixing the first rod, and then switching the other two rods.

Teacher:	Okay. What's the next three-car train that you can list? Let's try to list them in a way that help us know that we have all of them. Does someone have a good idea which one to do next? Remember the other day, we were looking, also I think, Thailee, you had a method the other day, didn't you? For listing them? What would be a good one to list next that help us when we're done to be sure that we had six? Eric, do you have an idea?
Eric:	Uhm, g plus p plus y.
Teacher:	Okay, so I'll write g plus p plus y. That is different one. Why would that help us, how would that help us know the end that we have all of them? Why did you list that one next?
Students:	(inaudible)
Teacher:	Sorry? (writing g+p+y on the board)
Eric:	Uhm…
Teacher:	Why did you pick that one?
Eric:	I want to say y plus g plus p.

Teacher: You wanna do y plu—you wanna change it?
Eric: Yeah.
Teacher: So, which one do you want to me list?
Eric: y plus g plus p.
Teacher: y plus g plus p? Why did you change it?
Eric: Uhm, because the first one, uhm, you can start with all y's, so first one, y plus p plus g, so you can reverse the p and g make it y plus g plus p.

Eric initially proposes gpy as the second list. The teacher points out that it is a different one, but keeps it recorded on the board. Instead of accepting the random pick, the teacher highlights the reasoning behind the selection. In responding to the teacher's request for explaining the reasoning behind his pick, Eric changes gpy to ygp. For the reason for changing the second solution to be listed, Eric provides an explanation for fixing the first letter and then switching the other two letters around.

Teacher: Do people agree with that that one is good to list as a second?
Students: Yes.
Teacher: Is this the method we used the other day, Remember? So can someone make another one that begins with y? So we can stick with y's?
Students: (some students raise their hands, but put their hands down)
Teacher: Why did you put your hands down, Macaulay?
Macaulay: Because you can't make more y's.
Teacher: You can't? How do you know that?
Macaulay: Because there's only three, three colors, and you can use only them two times.
Teacher: Can you say why you can only do two that have y in front?
Macaulay: Because you can't, you can't add two yellows, because if you do, it will be out of the order.
Teacher: Okay. Can anyone else explain besides—addition to what Macaulay said why we can't make another one starts with y with the conditions of this problem? Yeah, Terrence?
Terrence: Well, if uhm, if there's only two because p plus g and then g plus p, but there's only two and you do down like one more y, that's beginning with y, there's no more because you're trying p plus g or g plus p again.
Teacher: What do people think about what Terrence and Macaulay said? Do you agree or disagree with that? Zahara?
Zahara: I agree.
Teacher: Does anyone disagree with this idea? Dahlia, do you wanna add something?
Dahlia: I agree.

Teacher: You agree? Who has an idea of the next—oh, Thailee, do you wanna say something?
Thailee: Yeah. Uhm, yeah, I would like to say, a kind of what Terrence said, if you pretend like y isn't there, and there's just p and g, then you have to switch them around.
Teacher: Okay. So, if you pretend the y's not there, let me see if I can cover it, would you say that one more time so everyone look up what Thailee is saying that adds to what Macaulay and Terrence said? Okay, if y's not there—
Thailee: If you pretend that y isn't there, then there's just p and g and you switch them around.
Teacher: Do people see what she is saying? Mustafa, what she is saying?
Mustafa: [silent]
Teacher: (covering y's with a paper on the board) I'm covering up the y. What does she say about p and g?
Mustafa: [silent]
Teacher: Do you want to hear one more time?
Mustafa: (nodding his head) Uh-huh.
Teacher: Okay. (to Thailee) Say one more time?
Teacher: (to Mustafa) Listen carefully.
Thailee: Just, just switch the y-I mean p and g around.
Teacher: Can you say it now?
Mustafa: She said that you can switch y-p and g.
Teacher: How many times can you switch them around?
Thailee: [silent]
Teacher: How many times Thailee, can you switch them around?
Thailee: Once.

After Eric revises his suggestion for listing ygp as a second solution, the teacher seeks a collective agreement about the structure of listing. The teacher reminds students of the method used for a similar problem (arranging three-digit number using 1, 2, and 3) and then asks about the exhaustiveness of two solutions that begin with y. In the initial explanation, Macaulay addresses one of conditions of the problem (using three colors) and points out two solutions that begin with each color. In response to the teacher's request for repeating, Macaulay addresses another condition of the problem (using exactly once) and illustrates that one of three possible arrangements (i.e., y) violates the condition of the problem. Although Macaulay does not specify the reference of two times in his initial explanation and mixes the mundane meaning of "order" with the mathematical meaning of "order," he proves the exhaustiveness of two solutions with the given conditions of the problem. After Macaulay's explanation, the teacher seeks for additional explanation. Terrence explains two ways of arranging p and g and one more trial produces the repeated solution. Whereas Macaulay's explanation exhibits the exhaustiveness within the given conditions of the problem, Terrence's explanation focuses on the method of switching two rods (p and g) around. Instead of asking to repeat the explanation provided by either Macaulay or Terrence, the teacher asks whether other students agree or disagree with their explanations. Zahara and Dahlia express their agreements, but the teacher does not probe further the reason for agreeing but checks whether there is anything to add on the explanations. At the moment when the teacher moves to elicit a third solution to be listed, Thailee rephrases Terrence's explanation by constraining the context into arranging only two letters and adds the language of "switching" two letters around. After the teacher repeats Thailee's explanation, the teacher asks Mustafa to repeat Thailee's explanation. Mustafa was not able to repeat Thailee's explanation for the first time, so the teacher gives a turn to Thailee again to repeat her explanation. After Thailee's repetition, Mustafa repeats the idea of switching p and g around. Through the process of repeating or restating, the initial explanation becomes mathematically enriched:

- Macaulay's initial explanation addresses one of the conditions (using three rods) and two solutions with the same rod at the beginning
- Macaulay's repeated explanation addresses another condition of the problem (using exactly once) and illustrates the example which begins with the y but violates the condition
- Terrence's additional explanation addresses two ways of arranging p and g and another trial repeats one of solutions
- Zahara and Dahlia agree
- Thailee rephrases Terrence's explanation by constraining into arranging p and g and adding the language of switching them around.

After the class proves the exhaustiveness of two solutions that begins with y, the teacher continues to elicit a good candidate to be listed as a third solution. Jaclyn proposes gpy, but the teacher does not probe further why it is a good one to be listed as a third solution because a rigorous mathematical connection is not required for the third solution. After recording Jaclyn's proposal on the board, the teacher continues to elicit the next solution. For Madeline's proposal of gyp, the teacher probes further the reason for picking it.

Teacher:	[writing g+p+y on the board] Okay, who know what we want to list next, if you are trying to be strategic about how to list them? What would we list next? Madeline?
Madeline:	green plus yellow plus purple. (writing g+y+p on the board)
Teacher:	Why did you pick that, Madeline?
Madeline:	Because it's the opposite as the, uhm, it's the opposite-
Teacher:	-Say that again?
Madeline:	it's the opposite as green plus p plus yellow
Teacher:	How was it opposite?
Madeline:	Uhm, because the g is the same, except y and p, just turned around.
Teacher:	It uses the same front and y and p are turned around.

Madeline provides the rationale for listing gpy as a fourth solution. In re-explaining her reasoning, Madeline elaborates on the language from "opposite" to "turn around" and adds the same letter at the beginning. After repeating Madeline's explanation, the teacher continues to elicit the next solution. Jason proposes pgy as the fifth solution and Michael proposes pyg as the sixth solution.

The initial listed solutions
y+p+g (proposed by Ahmed)
y+g+p (proposed by Eric)
g+p+y (proposed by Jaclyn)
g+y+p (proposed by Madeline)
p+g+y (proposed by Jason)
p+y+g (proposed by Michael)

After eliciting all of the solutions, the teacher counts the number of solutions and then links it back to Thailee's conjecture. The teacher makes a space for students to talk with a partner. After providing one-minute short partner work, the discussion wraps up with briefly checking the degree to which students agreement on Thailee's conjecture.

The three-digit problem is introduced on Day 7 and the tree-car train problem is introduced on Day 9. The class collectively constructs an explanation for why they believe they have found all of the solutions.

- For the initial three-digit problem (1-2-3 problem), 312 is proposed by Kassandra, 321 is proposed by Ella, 132 is proposed by Ahmed, 231 is proposed by Hala, 213 is proposed by Macaulay, and 123 is proposed by Javonte.
- After eliciting six solutions, Jaclyn claims that there are no more answers.
- To support Jaclyn's claim, Devante shares his reasoning as repeating, Elias and Coretta repeat Devante's explanation, Devante confirms it, and the teacher restates it.
- To support Jaclyn's claim, Thailee suggest to putting them the solutions in a numerical order. Thailee reorders the listed solutions into123 and 132, Shelly reorders the listed solutions to 213 and 231, and Eric reorders the listed solutions into 312 and 321.
- After reorganizing the solutions, the class engages with the additional problem (3-5-7 problem). Bernard proposes the first pair of solutions (357 and 375) by using Thailee's method, Ahmed explains how Bernard's proposal is similar to Thailee's method, and Eric restates Bernard's proposal. Devante proposes the second pair of solutions (537 and 573) and Samara proposes the third pair of solutions (735 and 753).
- For the three-car train problem, Ella asserts that there are six solutions and Thailee makes a conjecture that there are six combinations to arrange three things. Thailee's assertion is repeated by Zahara, Thailee, and Ahmed.
- Ahmed proposes ygp, Eric proposes ygp, Jaclyn proposes gpy, Madeline proposes gyp, Jason proposes pgy, and Michael proposes pyg.

To support students construct a mathematical explanation collectively, the teacher provides the following instructional supports.

- On Day 7, for the three-digit problem, the key terms (e.g., digit; only once) is clarified and the conditions of the problem is elicited.
- Utilize the conditions of the problem to justify why the proposed solution is accepted as a correct answer
- Collect solutions from a group of students rather than taking up one individual student's complete idea in a public space

- Does not give a clue that all solutions are proposed, but prompts a serious consideration about the proposed solutions until figuring out that it is already listed
- Reconstructs the proposed solutions by applying the reasoning

7.4. The Case of EML 2013

Preview

In the EML 2013, the three-digit problem is introduced as a warm-up problem of Day 3. Like other warm-up problems, each student starts to work on the problem without any formal set-up stage. The whole-group discussion begins by eliciting incorrect answers for the problem and then extracts the conditions of the problem: (1) three-digit number; (2) using each digit only once; and (3) using 4, 5, and 6. Instead of taking up one individual student's complete list, the teacher elicits solutions from students one by one: 645 by Deshawn, 546 by Kallie, 456 by Aryanna, 654 by Otis; 465 by Ty and 564 by Tina. After eliciting all of six solutions, the teacher does not express that the class has found all of the solutions, but continuously seeks for other solutions. After the duplicated solutions were reviewed (654 by Kalvin and 564 by Demonte), Kallie claims that they have found all of the solutions.

Extensive Detailed Analysis

The three-digit problem is introduced in the first session of Day 3, assigned as a warm-up problem. Like other warm-up problems, each student starts to work on the warm-up problem of Day 3 after doing some daily routines without any formal set-up of the problem in a whole-group setting.

How many different three-digit numbers can you make using the digits 4, 5, and 6, and using each digit only once? Show all the three-digit numbers that you found. How do you know that you found them all?

During individual work, all except D'lon do not experience difficulties in understanding the problem. D'lon expresses his difficulty in understanding the problem but figures out what to do after reading aloud the problem to the teacher. After 13-minute individual work, the teacher convenes the class to discuss about the three-digit problem in a whole group. The teacher starts the discussion by asking students to read aloud the problem. After Aziz, Madeline, and Anaya read each sentence of the problem statement, the teacher elicits an example of incorrect answers for this problem.

Instead of eliciting correct answers to the problem, the teacher opens up the discussion by eliciting incorrect answers. Connor provides a general statement of using the same number twice then proposes that 665 is a wrong answer. The teacher seeks the agreement on the incorrect answer and asks others to explain why it is a wrong answer. Based on the incorrect answer that Connor proposed, Demonte extracts the first condition of the problem: using each digit only once. The teacher continues to collect different wrong answers to the problem. Liberty proposes 322 as a wrong answer and Kalvin explains that the proposed incorrect answer does not use any of the digits 4, 5, and 6. Ty proposes 4566 as a wrong answer. After seeking the agreement that 4566 is a wrong answer, David explains that the proposed incorrect answer has two 6s and has more than three digits. While these three incorrect answers are proposed, the teacher underlines the phrase in the problem statement and summarizes what the correct answer should meet.

- Three-digit number
- Using each digit only once
- Have to use 4, 5, and 6

The teacher then introduces a new word of conditions and explains what it means in mathematics.

Teacher: So, the conditions of a problem, what you just did, they are things that you have to make the right answer. That's how you can tell if your answer is right. If your answer meets all of the conditions of the problem, meets I mean, it does the three things, or something they might be more than three things, or less than three things, our problem has three conditions, and for you to do if your answer is right, you have to check three conditions. Okay, so I'm going to write down what this is and you can put it in your notebook. The conditions of a problem are the things that you have to do to get a right answer. The things that you have to do to get a right answer. So these things right here are all wrong answers because they didn't meet the conditions of a problem.

After identifying the three conditions of the problem, accompanied by explaining the meaning of conditions, the teacher elicits a correct answer to the problem with expressing the expectation of explaining the correct answer using the conditions of the problem.

Deshawn: Six, four, five?

Teacher:	Okay, six, four, five. Deshawn, do you think that you can explain why that's the right answer using the conditions?
Deshawn:	Uhm, it's the right answer because use three-digit number, four, five, six, and only use each digit one-only once.

Deshawn shares the correct answer and explains why it is correct using the conditions of the problem. After a quick check by asking about the completeness of his explanation, the teacher elicits the second answer.

Kallie:	Five, four, six.
Teacher:	Five, four, six. Kallie, do you think that you could explain it using conditions?
Kallie:	Well, what I got right is it has three digits in it, it only uses one of digits once in the problem.
Teacher:	If you use each digit only once and one more thing
Kallie:	And… it's using four, five, and six.
Teacher:	Good! It's using-it has three-digit, it doesn't use any number more than once, and it's using four, five, and six.

Kallie provides a complete explanation for why 564 is a correct answer to the problem. Aryanna gives 456 and the teacher asks D'lon to explain using the conditions of the problem. Then, Otis proposes another answer of 638.

Otis:	Six hundred thirty-eight.
Teacher:	Say that again?
Otis:	Six hundred thirty-eight.
Ahmed:	Thirty?
Otis:	Thirty-eight. Six hundred thirty-eight.
Teacher:	This one?
Student:	That's a wrong answer.
Ahmed:	That's wrong.
Teacher:	When someone gives an answer, what we always do is what Ahmed did yesterday, when he was teaching, what D'lon did, what did they ask every time when someone gives an answer? Tina, what did they say?
Tina:	How did get the answer?
Teacher:	How did get that? Every time someone gives an answer, you ask how did you get that. You don't say, uh. Okay, we gonna ask Otis how did you get that. Can you show the conditions and how that worked? What's the first condition of the problem?
Otis:	It's wrong.
Teacher:	It's wrong?
Otis:	Yeah.

Teacher:	You wanna change it?
Otis:	(nodding his head)
Teacher:	Okay. So you wanna give us-why is it wrong by the way?
Otis:	Because I'm not using four or five.
Teacher:	You're using the six, but you didn't use four and five. But you did make three-digit number. And you did use each digit only once. So, it's only this condition that you didn't follow. Right?
Otis:	(nodding his head)
Teacher:	Okay. You have the answer that would work?
Otis:	Six hundred fifty-four.
Teacher:	Okay. What did Otis-before we explain Otis's, what did Otis just do really good, when he put six hundred thirty-eight? What was really good when he started to talk about it? Can someone say what Otis did was good? That's part of being a really good at learning mathematics? Ahmed, what did he do?
Ahmed:	What do you mean?
Teacher:	What did Otis do was a good part of being a good learner in mathematics? What did he do? He put up six hundred thirty-eight, what did he do after that? Kallie?
Kallie:	He corrected himself.
Teacher:	He corrected himself and changed his mind because he was thinking about reasons.

When Otis proposes his answer of 638, a number of students immediately respond that it is a wrong answer. The teacher uses this moment to remind norms about how to respond to the incorrect answer and then asks how to get the answer by checking the conditions of the problem. When the teacher asks Otis how the proposed answer meets the condition, he admits that his proposed answer is wrong because it does not meet the condition of the problem and then suggest another answer of 654. After commenting what Otis did well, the teacher gives Otis another turn to explain his new proposal using the conditions of the problem. Next, Ty gives an answer of 465 and April explains how it meets the conditions of the problem. After seeking a collective agreement, the teacher gives a turn to Tina. Tina suggests 564 and Tenisha explains the conditions of the problem with the teacher's support.

Without expressing that all of the solutions are proposed, the teacher continues to collect another solution. Kalvin proposes 654 and the teacher points out that it is already listed on the board. The teacher asks students to look at their notebooks and to propose another answer. Demonte proposed 564 but the teacher points out that it is already listed. At this point, Kallie claims that they found all of the solutions.

Kallie:	Uhm, I believe that they are all done.
Student:	[Six hundred fifty-four?]
Kallie:	Because I remember each-the first digit like six is only supposed to have, six is only supposed-six starts two numbers, five starts two numbers, and four starts two numbers.
Teacher:	Kallie is saying that this is all that we can make. Could you turn and talk to the person next to you and see if that's right or if we're missing any? Find if a person next to you agrees.

Kallie finds the structure of six solutions that are listed on the board. She notices that two of them start with 4 in the hundreds place, two of them start with 5 in the hundreds place, and two of them start with 6 in the hundreds place. After repeating Kallie's assertion that they found all of them but not repeating the structure, the teacher asks students to talk with a partner whether Kallie's assertion is right and whether there are any missing solutions. During partner work, the teacher asks April whether they found all of the solutions.

Teacher:	What do you think? Do you think that this is all they are?
April:	Yeah.
Teacher:	How do you know?
April:	Because what I did, I put four two times, and I put two [inaudible] two times, and all the other numbers it has and I just changed the last two numbers. So four, five, six and four, six, five.
Teacher:	So, you're pretty sure.
April:	(nodding her head)

Like Kallie, April develops the structure of her solutions, by pairing two solutions that have the same number in the hundreds place, but then adds that the last two digits are switched around. After a short partner work, the teacher reconvenes the class again to discuss. Instead of highlighting the structure Kallie found, the teacher puts more emphasis on Kallie's idea that they found all of the solutions and on hearing others' thinking.

Teacher:	Okay, I would like to hear what people think right now. So, Kallie is saying, Kallie's idea we found all of them, and she tried out to

	explain why she thinks so. But I would like to know more people's thinking. Do you think that this is all the numbers that we can find from this problem?
Students:	Yes.
Teacher:	Elysa, what do you think so? Shh. Hands down for a moment.
Elysa:	Uhm… because-
Teacher:	Why do you think so, Elysa?
Elysa:	They are only supposed to be…
Teacher:	It's hard to hear Elysa right now, could everybody's eyes on Elysa and be listening? Speak up a little bit, Elysa. Why do you think that these are all we can make?
Elysa:	Because some of them are like backwards.
Teacher:	Some of them are like backwards? Can you give me an example of something backwards?
Elysa:	Uhm….
Teacher:	What do you see up here where-what you mean by backwards?
Elysa:	Uhm, four hundred sixty-five and four hundred fifty-six.
Teacher:	Four hundred sixty-five and what?
Elysa:	Four hundred and fifty-six.
Teacher:	Okay, so could everyone look up what Elysa's showing right now? She's calling this backwards. Can you explain what you mean by backwards here?
Elysa:	You have to turn it around and make other number.

Both Kallie and Elysa notice the structure of the six solutions and group two solutions together. One difference would be that Kallie focuses on the first digit, whereas Elysa focuses on the last two digits. When Elysa names the pattern as backwards, the teacher seeks for an example of backwards. As an example of backwards, Elysa illustrates 465 and 465. Tenisha and Ahmed add onto Elysa's explanation.

Teacher:	So could someone add onto what Elysa said? That's really helpful, Elysa. What is Elysa showing about these two? That's very important to solve this problem. Tenisha, what do you see here?
Tenisha:	Uhm, well, there's still four there, but this is switching five and six.
Teacher:	Good. Four is still in the front, and we're switching five and six. So, can we put four in the front again?
Students:	No.
Teacher:	Why we can't put four in front again? Ahmed?
Ahmed:	Because that would mean it's not backwards.
Teacher:	What would be happened if we try to put four in front again?
Ahmed:	If you put four in front, that would be the same hundred.
Teacher:	Does everyone agree that if we put four in front again, we will get the same number again?

Students:	Yes.

Tenisha adds that the digit in the hundreds place is the same but the digit in the tens place and the digit in the ones place are switched around. In fact, Tenisha's explanation combines Kallie's observation and Elysa's observations. After repeating Tenisha's explanation, the teacher asks about the possibility of another number with the digit of 4 in the hundreds place. By asking the exhaustiveness of solutions with the digit 4 in the hundreds place, the teacher asks for proving all of the solutions in a simplified version but maintains the same mathematical demand. For the reason why they cannot put four in front again, Ahmed rationalizes that not having four in the front is not a backward of 456 or 465. In responding to Ahmed's explanation for the solutions which do not have the digit of 4 in the hundreds place, the teacher asks again about the case if putting four in the hundreds place again. Even though Ahmed's response is not complete, the teacher asks for an agreement that putting 4 in the hundreds place repeats the number and asks the agreement on the exhaustiveness of solutions in which have 4 in the hundreds place. Through this process, the teacher supports students to build the explanation collectively.

After completing the explanation about the exhaustiveness of solutions for 4 in the hundreds place, the teacher distributes an opportunity to apply the reasoning that was constructed for the case of 4 in the hundreds place. After the teacher debriefs Elysa's method (fixing five in the hundreds place and switching around the digit in the tens place with the digit in the ones place), Jarvaise nominates two solutions: 546 and 564. The teacher gives April a turn to nominate two solutions that have 6 in the hundreds place and Elysa's method works. In reviewing how Elysa's method works, the class reorganizes the previous solutions with the same hundreds (465 and 456; 546 and 564; and 654 and 645) and shares the opportunity to apply Elysa's method. At this point, the teacher moves from applying Elysa's method in the simplified version to synthesizing how Elysa's method works to explain that they are done. Tina gets a turn.

Tina:	So, how Kallie says, it could be only two fours and how Elysa says they're backwards-
Teacher:	-okay-
Tina:	-so you're saying first four, six, five and four, five, six and you do it again and go five, four, six and five, six, four and then six, four, five, and six, five, four. So, it's like switching like flip flaps.

Teacher: So I would like to have somebody else explaining it. Like to show that you're really listening to Elysa and Tina talked and I would like to someone to explain what they are showing. This is a way to show you found all of the answers. Can someone else explain it? Okay, Kalvin, what would you say?

Kalvin: Uhm, that-

Teacher: Set up your hands away from your mouth when you are talking. Okay.

Kalvin: They're saying it's going backwards because it-they're saying, Tina and Elysa are saying that it changes every time, it like-like Tina said like four, six, five and then, and then four, five, six and then five, four, six and five, six, four and then six, five, four and six, four, five.

In synthesizing the idea of paring up two solutions that have the same digit in the hundreds place with the idea of the pattern of switching the last two digits, Tina makes references to other students (Kallie and Elysa) and makes the structure explicit. Kalvin also refers to other students (Tina and Elysa) and makes the structure visible by pairing two solutions together. The teacher then asks students to write in their notebooks how many answers they found and how they know that they found all of them.

The class collectively constructs an explanation for how they have found all of the solutions.

- Kallie shares her initial observation about paring numbers that have the same digit in the hundreds place;
- Elysa notices the "backwards" pattern which involves switching around the digit in the tens place with the digit in the ones place;
- Tenisha combines Kallie's observation and Elysa's observation;
- Ahmed adds the repetition;
- A collective agreement is made on the exhaustiveness of solutions for 4 in the hundreds place.
- Jarvaise applies Elysa's method for the cases which have 5 in the hundreds place
- April applies Elysa's method for the cases which have 6 in the hundreds place.
- Tina makes references to Kallie and Elysa and pairs up solutions.
- Kalvin makes references to Tina and Elysa and pairs up solutions.

To support students in constructing a mathematical explanation collectively, the teacher provides the following instructional supports.

- Elicits incorrect answers to extract the conditions of the problem
- Utilizes the conditions of the problem to justify why a proposed solution is accepted as a correct answer
- Does not immediately reject the incorrect answer, but examines the reasoning and revising one's idea
- Collects solutions from a group of students rather than taking up one individual student's idea in front
- Does not give a clue that all solutions are proposed, but offers a serious consideration about the proposed solutions until figuring out that it is already listed
- Provides time to think about the exhaustiveness of the solutions before parsing out the reason behind the exhaustiveness
- Applies one's reasoning and re-structures the proposed solution
- Makes a reference to an established idea and seeks a collective agreement at the critical moments.

7.5. Summary of the Chapter

In Chapter 7, I analyzed instructional interactions managed by the teacher, Ms. Ball, for teaching the three-permutation problem across three years (EML 2009, EML 2010, and EML 2013). This section summarizes the elements of mathematical explanation for the three-permutation problem, the problems that students have in explaining the three-permutation problem and then features of instructional supports to resolve such problems.

The three-permutation problem involves producing multiple solutions that meet the conditions of the problem and providing the exhaustiveness of solutions. The mathematical explanation for the three-permutation problem involves the following elements:

- Explaining why it is acceptable as a correct answer for each proposal, which includes (1) using three things and (2) using exactly once. Any violation of these two conditions could not be accepted as an answer for the permutation problem
- The repetition of the same thing is not allowed (exactly once)
- Justifying that the order matters
- Grounding on the mathematical structure rather than empirical experiences

Building on the features of students' explanation provided at the end of individual year analysis in Chapter 7.2, this section provides a more general characterization and a more comprehensive collection of possible problems that individual students have in offering an explanation for the three-permutation problem.

- Difficulties with accessing to the entry point of the mathematical task: A lack of understanding about what the three-permutation problem is asking is an obstacle to accessing to the entry point of the mathematical task.
- Relying on the empirical experiences
- Being confused with the previous mathematical task (e.g., "Train problem part 1")
- Having a lack of system to build a structure of solutions
- Having less issues with inaccurate, incorrect, implicit, and vague language use

To support the students' development of mathematical explanation, the teacher provides the following instructional supports:

- Providing more extensive instructional supports at the beginning of the lesson, compared to the instructional supports provided for the brown rectangle problem (see Chapter 4) and the blue and green rectangle problem (see Chapter 5)
- Eliciting examples or non-examples to identify the conditions of the problem and to clarify the problem statement
- Eliminating incorrect answers at the beginning of the lesson
- Eliciting multiple solutions one by one instead of taking up one individual student's complete list
- Providing exposure or notice to the structure of solutions
- Having more control of the way in which a public recording is made on the board

CHAPTER 8.
SUPPORTING STUDENTS TO DEVELOP MATHEMATICAL EXPLANATION: FINDINGS FROM ACROSS YEARS AND ACROSS MATHEMATICAL TASKS

8.1. Overview

In the previous four chapters (Chapters 4, 5, 6, and 7), I analyzed instructional interactions managed by the same teacher for teaching the same mathematical tasks to different cohorts of students. Building on the individual year analyses, this chapter discusses issues that arise from the analysis of developing mathematical explanation for the same mathematical tasks across years and from the analysis of developing mathematical explanation across multiple mathematical tasks.

In this dissertation study, it is crucial to look at the multiple uses of the same mathematical tasks by the same teacher across different cohorts of students. An in-depth analysis of a single case of teaching might provide rich information about features of students' mathematical explanation, both individually and collectively, and instructional supports that the teacher provides for the development of students' mathematical explanation. Despite its potential richness, a single case study of teaching is not sufficient for this dissertation study. First, a single case study of teaching might be limited by describing what a particular teacher knows, does, or says at a particular moment rather than revealing the underlying structure that frames the work of teaching in the dynamic of instructional interactions. Given that this dissertation studies teaching rather than a teacher, the multiple cases of teaching would better serve for revealing the variety of issues that can arise in teaching and for framing the underlying structure to deal with such various issues. Second, as discussed in Chapter 2, the cross-year analysis, especially when holding other key variables of instructions (teacher and content) relatively constant but only varying students, provides a useful analytical tool to unravel one of the greatest predicaments of teaching—its dependence on students.

Another feature of this dissertation study is to look at the use of multiple mathematical tasks by the same teacher. Given that the ideas around leading a whole-group discussion and creating a mathematical discourse community are more generic but are not much differentiated by different mathematical tasks in the literature, it is worthwhile to examine whether difficulties that students have in constructing mathematical explanation might be different from one mathematical task to another mathematical task. If it is the case, the level of mathematical supports and the use of instructional resources to support the work might differ across mathematical tasks.

In the following sections, I discuss the findings from the analysis of developing mathematical explanation for the same mathematical tasks across years and the findings from the analysis of developing mathematical explanation across mathematical tasks. These findings provide a basis for eliciting the demands entailed in the work of supporting students to develop mathematical explanation and framing the underlying structure that serves to meet such demands.

8.2. Supporting Students To Develop Mathematical Explanation: Findings From the Cross-Year Analysis

Using the data yielded from the individual year analyses in the previous four chapters, this section first examines what kinds of difference exist in teaching the same mathematical task even by the same teacher and then explores whether or not such differences matter for the collective construction of mathematical explanation and if so in what ways.

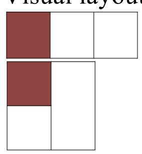
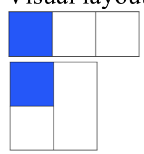
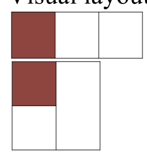
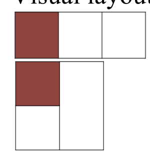
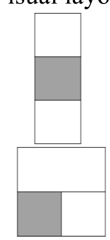
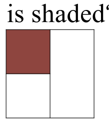
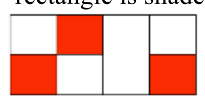
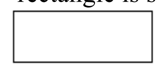
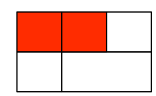
The data analyses in the previous chapters illustrate that instruction for teaching the same mathematical task unfolds somewhat differently, even by the same teacher. The similarities across multiple cases become strong candidates to be scaled up into the coherent structure of supporting students to develop mathematical explanation, whereas the differences across multiple cases offer analytical opportunities to examine whether or not the particular instructional feature plays a role in supporting students to develop mathematical explanation. This section mainly discusses what such differences reveal about the work of supporting students to develop mathematical explanation. In doing so, I do not treat such differences as discrepant or disconfirming evidence, but use the differences as the data to reveal the underlying structure of the work of teaching entailed in supporting students to develop mathematical explanation. In addition, the differences observed across years do not necessarily represent the characteristics of the expert teacher's teaching practice.

One of the possible reasons for such differences might be that the EML is public teaching in which a number of pedagogical and mathematical ideas are tried, tested, and examined by the EML observers' suggestions during pre-briefing or de-briefing session. Another reason might be that the teacher might reflect on the previous years' instructional interactions and make some amendments in the subsequent years. Lastly, as discussed in Chapter 2, similar to other professions which aim for human improvement, teaching depends on its clients (i.e., students), thus it is no wonder that the unfolding instructions differ due to the mathematical ideas, stances, dispositions, and issues that students bring each year. However, investigating the reason for such differences, inferring the teacher's intention of adopting a particular approach, or evaluating the most effective approach is beyond the scope of this dissertation study.

The differences could be analyzed in two ways: (1) differences that arise due to variables that a teacher cannot control such as instructional contexts and students and (2) differences that are caused by variables for which a teacher makes deliberate choices such as pedagogical and mathematical approaches. For the first type of variables that a teacher cannot control, I examine whether or not they impose different demands to support students to develop mathematical explanation; if so, I examine how to adjust the work to meet such demands to achieve the same instructional goal of supporting students to develop mathematical explanation. For the second type of variables for which a teacher makes deliberate choices, I examine how a particular pedagogical or mathematical approach might interact with the collective construction of mathematical explanation. In comparing pedagogical or mathematical approaches, I do not evaluate the effectiveness of a particular approach, do not propose that a particular approach should be adopted by a teacher, or do not make a causal claim that a particular approach always produces the collective construction of mathematical explanation in a particular way.

Tables 8.1, 8.2, 8.3, and 8.4 compare instructional features that seem to be relevant in the collective construction of mathematical explanation for the same mathematical tasks. As shown in the Tables, in teaching the same mathematical tasks, some instructional features (e.g., what kinds of mathematical supports are provided while launching a mathematical task; what kinds of mathematical supports are provided while students are working independently) are not much differentiated across years, but others (e.g., time point; selecting; sequencing; the proportion of correct answers to incorrect answers) are quite different from one year to another. This section does not exhaustively analyze every feature listed in the Tables 8.1 through 8.4, but provides analyses of some aspects which have been considered as critical factors in developing mathematical explanations in the literature and some aspects which show interesting interactions with the collective construction of mathematical explanation. Following the Tables, I discuss whether or not such differences in (1) instructional contexts; (2) pedagogical approaches; (3) mathematical approaches; and (4) students' mathematical ideas, stances, dispositions, and issues might matter for the collective construction of mathematical explanation and if so how.

Table 8.1. The comparison of interactional features for teaching the brown rectangle problem

	EML2007	EML2008	EML2009	EML2010	EML2013
Day (Time)	• Day 6 (30 minutes)	• Day 1 (20 minutes)	• Day 4 (24 minutes)	• Day 2 (35 minutes) • Day 6 (18 minutes)	• Day 1 (20 minutes) • Day 2 (20 minutes)
Homework	• Assigned as homework in the previous week	• Not assigned as homework previously	• Not assigned as homework previously	• Not assigned as homework previously	• Not assigned as homework previously
Presentation of the task	• Part A and Part B are separately posted • Problem statement (not written on the poster): "What fraction of ***the big rectangle*** is shaded brown?" • Visual layout  • Drawn on the non-grid poster	• Part A and Part B are posted together • Problem statement (written on the poster): "What fraction of ***each rectangle*** below is shaded?" • Visual layout  • Drawn on the grid-poster	• Part A and Part B are separately posted • Problem statement (written on the poster): "What fraction of ***the rectangle*** below is shaded in?" • Visual layout  • Drawn on the non-grid poster	• Part A and Part B are posted together • Problem statement (written on the poster): "What fraction of ***the rectangle*** is shaded brown?" • Visual layout  • Drawn on the non-grid poster	• Part A and Part B are posted together • Problem statement (written on the poster): "What fraction of ***the rectangle*** is shaded gray?" • Visual layout  • Drawn on the non-grid poster
Additional task used	• Improvised • If the brown rectangle is the whole, how much is shaded? 	• None	• Planned in advance • What fraction of the rectangle is shaded red? 	• Improvised • Draw 1/5 of the rectangle is shaded 	• Planned in advance • What fraction of the rectangle below is shaded red? 
Launching the task	• Not ask students to read aloud the problem statement	• Ask a student to read aloud the problem statement	• Not ask students to read aloud the problem statement	• Ask a student to read aloud the problem statement	• Not ask students to read aloud the problem statement

	• Not specify what "the big rectangle" refers to • Share an observation that most students get the same answer for the first problem but different answers for the second problem	• Not specify what "rectangle" refers to • Share an observation that most students get the same answer for the first problem but different answers for the second problem	• Not specify what "the rectangle" refers to • Not share an observation about the answers that the students produce	• Not specify what "the rectangle" refers to • Not share an observation about the answers that the students produce	• Not specify what "the rectangle" refers to • Share an observation of seeing different answers, but does not specify for which problem
Individual /Partner work	• Individual work	• Individual work	• Individual work → Partner work	• Individual Work	• Individual work
Circulating during individual /partner work	• Check students' work (whether students write down a number or not) but do not provide any substantive mathematical supports spontaneously • No mathematical requests from students	• Check students' work but do not provide substantive mathematical supports spontaneously unless the requests are made by students • Students request help for clarifying mathematical issues but those issues are resolved in a private space	• Check student's work (answer and reason) but do not provide substantive mathematical supports spontaneously • No mathematical requests from students	• Check students' work (answer and reason) but do not provide substantive mathematical supports spontaneously unless the requests are made by students • One student makes a request for help, but the teacher resolves the issues in a private space	• Check students' work (mainly answer) but do not provide substantive mathematical supports spontaneously unless the request are made by students • One student makes a request for help, but the teacher responds that they will discuss the problem soon
Production of written explanation	• No	• No	• Yes	• Yes	• No
Written answers in the notebook	• First problem: 1/3 • Second problem: 1/4, 1/3, 1/2, $1\frac{1}{3}$	• First problem: 1/3, 1/2, 2/6, 3/9, 5/15 • Second problem: 1/4, 1/3, 1/2	• First problem: 1/3 • Second problem: 1/4	• First problem: 1/3, 1/2, 3/1 • Second problem: 1/4, 1/3, 4/1	• First problem: 1/3, 1/2 • Second problem: 1/4, 1/3, 1/2, $1\frac{1}{2}$
The proportion of answers	• First problem (equally partitioned): All of the students get the correct answer	• First problem (equally partitioned): Most of the students get the correct answer (23/26)	• First problem (equally partitioned): All of the students get the correct answer	• First problem (equally partitioned): Most of the students get the correct answer	• First problem (equally partitioned): Most of the students get the correct answer (23/27)

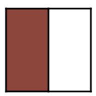
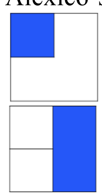
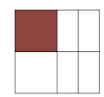
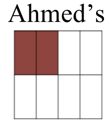
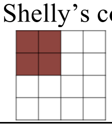
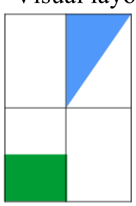
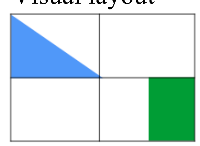
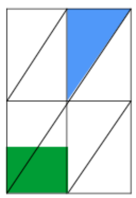
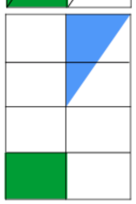
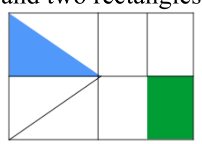
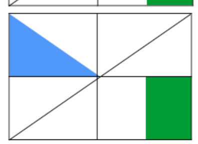
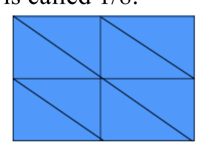
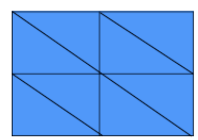
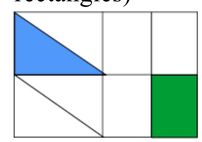
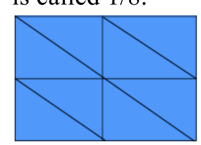
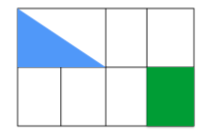
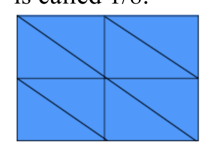
	• Second problem (unequally partitioned): ○ The proportion of students who record 1/3: 21/27 ○ The proportion of students who record 1/4: 6/27	• Second problem (unequally partitioned): ○ The proportion of students who record 1/3: (unavailable) • The proportion of students who record 1/4: 9/26	• Second problem (unequally partitioned): ○ The proportion of students who record 1/3: (unavailable) • The proportion of students who record 1/4: 23/25	• Second problem (unequally partitioned): ○ The proportion of students who record 1/3: 9/28 • The proportion of students who record 1/4: 17/28	• Second problem (unequally partitioned): ○ The proportion of students who record 1/3: 14/29 • The proportion of students who record 1/4: 12/29
The sequence of the proposed answers	• First problem: 1/3 • Second problem: 1/4 → $1\frac{1}{3}$ (nominated by the teacher) → 1/3→1/2	• First problem: 1/3 • Second problem: 1/3→1/4	• First problem: 1/3→ 2/3→2/6 • Second problem: 1/4→1/3 (introduced by the teacher)	• First problem: 1/3 • Second problem: 1/4→1/3→1/6→2/8→ 4/16	• First problem: 1/3→1/2 • Second problem: Not a fraction →1/4→$1\frac{1}{2}$
Alteration of the original mathematical tasks by the students	• Kurtis's comment 	• Alexico's comment 	• N/A	• Jaclyn's comment  • Coretta's comment  • Ahmed's comment  • Shelly's comment 	• N/A
Available resources	• N/A	• Grids on the poster	• N/A	• Sticky lines • Sticky shaded rectangles	• Sticky lines • Sticky shaded rectangles
Knowledge established at the end of lesson	• Making equal parts • Identifying the whole	• Making equal parts	• Making equal parts	• Making equal parts	• Identify the whole • Make equal parts • Name one of equal parts

Table 8.2. The comparison of interactional features for teaching the blue and green rectangle problem

	EML2007	EML2008	EML2009	EML2010	EML2013
Day (Time)	• Day 7 (33 minutes)	• Day 3 (32 minutes) • Day 4 (15 minutes)	• Day 5 (24 minutes) • Day 6 (51 minutes)	• Day 3 (60 minutes)	• Day 3 (21 minutes) • Day 4 (27 minutes)
Presentation of the task	• Problem statement (not written on the poster) o What fraction of the big rectangle is shaded blue? o What fraction of the big rectangle is shaded green? o How much of the big rectangle is shaded all together? • Visual layout  • Drawn on the non-grid poster	• Problem statement (written on the poster) o What fraction of the big rectangle is blue? o What fraction of the big rectangle is green? • Visual layout  • Drawn on the grid poster	• Problem statement (the first two are written on the poster from the beginning but the last one is verbally provided at the end) o What fraction of the big rectangle is the blue region? o What fraction of the big rectangle is the green region? o Are the green and the blue really the same area? • Visual layout  • Drawn on the non-grid poster	• Problem statement (the first two are written on the poster from the beginning and the last one is added on the board toward the end) o What fraction of the big rectangle is shaded green? o What fraction of the big rectangle is shaded blue? o How could the blue triangle and the green rectangle each be one-eighth of the big rectangle? Even though they are not the same shape? Can you prove it? • Visual layout  • Drawn on the non-grid poster	• Problem statement (the first two are written on the poster from the beginning but the last one is verbally provided at the end) o What fraction of the big rectangle is shaded green? o What fraction of the big rectangle is shaded blue? o Are they the same are or are they not the same area? • Visual layout  • Drawn on the non-grid poster
Launching the task	• Not clarify "the big	• Very explicitly clarify	• Briefly clarify the whole	• Briefly clarify the whole	• Not clarify "the big

	rectangle" during launching the task	what "the big rectangle" means while launching the problem	by tracing the big rectangle once but not in an extensive way	by tracing the big rectangle once but not in an extensive way	rectangle during launching the task
Individual/ Partner work	• Individual work	• Partner work	• Individual or partner work	• Partner work	• Individual or partner work
Circulating during individual/ partner work	• Provide mathematical supports for individual students upon a request in a private space • Remediate the incorrect identification about the whole	• Provide mathematical supports for individual students upon a request in a private space • Remediate the incorrect identification about the whole • Provide mathematical support not to deviate from the key ideas (only counting one color for the shaded parts; not excluding the other color from the whole) • Further challenge the same answer for the twodifferent shapes	• Provide mathematical supports for individual students upon a request in a private space	• Provide mathematical supports for individual students upon a request in a private space • Remind the students to use the working definition of fraction written on the board • Check the partner work	• Provide mathematical supports for individual students upon a request in a private space • Remind the students to use the working definition of fraction written on the board
Production of written explanation	• Yes	• No	• Yes	• Yes	• Yes
The availability of the working definition of fraction	• Yes, written on the poster 1. Identify the whole 2. Equal parts 3. How many parts of the whole	• Verbally mentioned about making equal parts • d of 1/d is written on the board	• Yes, written on the poster 1. Take some whole and divide into equal parts 2. One part is called 1/number of equal parts 3. The number of equals parts	• Yes, written on the poster 1. Identify the whole 2. Make the equal parts	• Yes, written on the poster 1. Figure out what the whole is. 2. Make sure that the whole is divided into equal parts. 3. Count how many equal parts are there. 4. Write 1/d to show

					one of the equal parts. 5. If more than 1 of those parts is shaded, count them (n) and write n/d.
Written answers in the notebook	• All but one student recorded the same answer for the blue triangle and for the green rectangle • Both for the blue triangle and the green rectangle: 1/8, 1/4 in half, $\frac{1/2}{4}$, 1/2, 1/4, 4/8, $1\frac{1}{2}$, $3\frac{1}{2}$	• All students recorded the same answer for the blue triangle and for the green rectangle. • Both for the blue triangle and the green rectangle: 1/8, half of a quarter, 1/2 of 1/4, 1/2+1/4=1/8, 1/2, 1/4, 2/4	• All students recorded the same answer for the blue triangle and for the green rectangle • Both for the blue triangle and the green rectangle: 1/8, 1/2 of 1/4, 1/2, 1/4, 2/4, 8/4, 2/6	• All students recorded the same answer for the blue triangle and for the green rectangle. • Both for the blue triangle and the green rectangle: 1/8, 1/4, 1/2, 1/6, 1/5	• Five students recorded different answers for the blue triangle and for the green rectangle. • Both for the blue triangle and the green rectangle: 1/8, 1/4, 2/8, 1/2, 1/6, 8/8
The proportion of answers	• The proportion of correct answers: 9/27 • The proportion of incorrect answers by not taking the intended whole: 10/27 (8 students wrote 1/2 and 2 students wrote 1/4)	• The proportion of correct answers: 17/26 • The proportion of incorrect answers by not taking the intended whole: 1/26 (one student wrote 1/2)	• The proportion of correct answers: 14/25 • The proportion of incorrect answers by not taking the intended whole: 4/25 (one student wrote 1/2 and three students wrote 1/4)	• The proportion of correct answers: 15/28 • The proportion of incorrect answers by not taking the intended whole: 9/28 (six students wrote 1/4 and three students wrote 1/2)	• The proportion of correct answers: 17/29 • The proportion of incorrect answers by not taking the intended whole: 2/29 (two students wrote 1/4)
The sequence of proposed answers	• For the blue triangle: 1/2→1/5→1/8 • For the green rectangle: 1/8→1/2	• For the blue triangle: 1/2 of 1/4→1/8→ 1/2 +1/4=1/8→1/2 (introduced by the teacher) →1/4 (introduced by the teacher) • For the green rectangle: 1/8	• For the green rectangle: 1/8→1/2 • For the blue triangle: 1/8	• For the green rectangle: 1/4 →1/8 • For the blue triangle: 1/8	• For the green rectangle: 1/8 • For the blue triangle:1/8

Practice of naming a fraction	• N/A	• N/A	• Practice with naming a fraction for more than one shaded parts	• N/A	• Practice with naming a fraction for more than one shaded parts
Issues that students have	• Dissecting the whole into the same shapes • Incorrectly converting “a fraction of a fraction” to a single fraction (e.g., 1/2 of 1/4 =1/5)	• Dissecting the whole into different shapes • Incorrectly converting “a fraction of a fraction” to a single fraction (e.g., 1/2 + 1/4 = 1/8)	• N/A	• Dissecting the whole into different shapes (Bernard’s challenge to the method proposed by Madeline and Chanika)	• Dissecting the whole into different shapes (Renee’s challenge to the method proposed by Tenisha)
Methods for proving	• The big rectangle is dissected into the same shapes (8 equal triangles for the blue triangle and 8 equal rectangles for the green rectangle) within the intended reference of the whole.  	• The big rectangle is not dissected into all the same shapes (four triangles and four rectangles; six triangles and two rectangles)   • It takes eight cutouts to cover the whole, so one is called 1/8. 	• It takes eight cutouts to cover the whole, so one is called 1/8. 	• The big rectangle is not dissected into all the same shapes (four triangles and four rectangles)  • It takes eight cutouts to cover the whole, so one is called 1/8. 	• The big rectangle is not dissected into all the same shapes (six rectangles and two triangles)  • It takes eight cutouts to cover the whole, so one is called 1/8. 

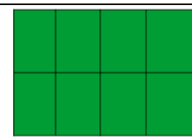
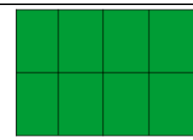
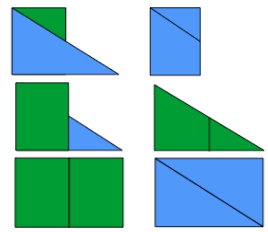
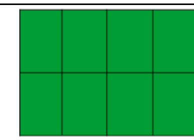
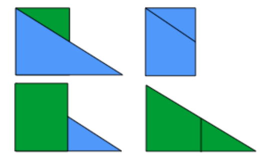
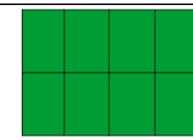
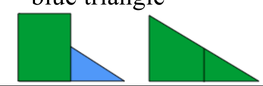
			 • Further challenge why the blue triangle and the green rectangle are the same 	 • Further challenge why the blue triangle and the green rectangle are the same 	 • Further challenge why the blue triangle and the green rectangle are the same o Kallie's method: measuring o Ahmed's method: 8 pieces make the whole o Otis's method: transforming the green rectangle to the blue triangle 
Available resources	• No	• Sticky blue triangles • Sticky green rectangles	• Sticky blue triangles • Sticky green rectangles • Scissor	• Sticky blue triangles • Sticky green rectangles • Scissor	• Sticky blue triangles • Sticky green rectangles • Scissor
Knowledge established at the end of lesson	1. Identify the whole 2. Equal parts 3. How many equal parts out of the whole		1. Identify the whole 2. Make d equal parts 3. Write 1/d to show one of the equal parts 4. If you have d of 1/d, then you have the whole. 5. If you have n of 1/d, then you write n/d. n and d are whole numbers for now (i.e., in fifth grade) • d≠0 (d cannot be 0)	1. Identify the whole 2. To identify the fraction, divide the whole into equal parts 3. When the whole is divided into d equal parts, we call one of the equal parts 1/d. 4. When we have all the equal parts, it is the whole, and we write d/d.	

Table 8.3. The comparison of interactional features for teaching the two-coin problem

	EML2010	**EML2013**
Day	• Day 3, Day 4	• Day 4
Instructional sequence	• The two-coin problem (Day 3, 4)→ The three-permutation problem (Day 7, 9)	• The three-permutation Problem (Day 3) → The two-coin problem (Day 4)
Presentation of the task	• Written problem statement on the board: I have pennies, nickels, and dimes in my pocket. If I pull out 2 coins, what amounts of money might I have? Prove that you have found all of the amounts possible.	• Written problem statement on the board: I have pennies, nickels, and dimes in my pocket. If I pull out 2 coins, what amounts of money might I have? Prove that you have found all of the amount possible.
Launching the task	• Read aloud the problem statement • Restate what the problem is asking	• Read aloud the problem statement • Restate what the problem is asking • Elicit one example and identify the conditions of the problem • Add examples and non-examples on the board
Individual/ Partner work	• Partner work	• Individual work
Circulating during individual/ partner work	• Understanding the problem • Figuring out the way to keep track of solutions • Clarifying the number of coins to pull out and the equal conditions among students • Eliminating the duplicated amounts • Spot the mathematical issues that boosts up a whole-group discussion	• Understanding the problem • Clarifying equal number of pennies, nickels, and dimes in the pocket • Clarifying the meaning of mathematical terms • Preventing further production of incorrect solutions which violate the conditions of the problem
The proportion of answers	• The proportion of students who produce all of six solutions: 18/28 • The proportion of students who violate the conditions of the problem: 0/28 • The proportion of students who duplicate the solutions with the same order: 6/28 • The proportion of students who duplicate the solutions with the different order: 0/28	• The proportion of students who produce all of six solutions: 15/29 • The proportion of students who violate the conditions of the problem: 9/28 • The proportion of students who duplicate the solutions with the same order: 0/28 • The proportion of students who duplicate the solutions with the different order: 4/28
Available resources	• Distributing a bag of coins	• Not distributing a bag of coins
Issues that students have	• Duplicating solutions, but not violating the conditions of problem • Having an issue whether a different order matters	• Violating the conditions of problem, but not duplicating solutions • Not having an issue whether a different order matters

Table 8.4. The comparison of interactional features for teaching the three-permutation problem

	EML2009	EML2010	EML2013
Day	• Day 4 (three-car train) • Day 5 (three-kids race) • Day 9 (three-car train)	• Day 7 (thee-digit number) • Day 9 (three-car train)	• Day 3 (three-digit number)
Instructional sequence	• Train Problem Part 1	• Train Problem Part 1 • The two-coin problem (Day 3, 4)→ The three-permutation problem (Day 7, 9)	• The three-permutation Problem (Day 3) → The two-coin problem (Day 4)
Presentation of the task	• The three-car train (red, light green, purple): Find all the ways to make different trains using exactly one of the red, light green, and purple rods. Keep track of each train. How many are there? How do you know you made all the possible trains using just those 3 rods? • The three-kids race: Three kids ran a race. James, Tasha, and Maria. We don't know the results. We just know that one person finished first, someone finished second, and someone finished third. Make a list of all of the possible results of the race. How do you know that you found all of the possibilities? • The three-car train (yellow, light green, purple): Find all the ways to arrange the light green, purple, and yellow rods into three car trains using exactly one of each rod. How are you sure you have found all the ways?	• The three-digit number: How many different three-digit numbers can you make using the digits 1, 2, and 3, and using each digit only once? Show all the three-digit numbers that you found. How do you know that you found them all? • The three-car train: Find all the ways to make different trains using exactly one each of the red, light green, and purple rods. Keep track of each train. How many are there? How do you know you have all the possible train using just those three rods?	• The three-digit number: How many different three-digit numbers can you make using the digits 4, 5, and 6, and using each digit only once? Show all the three-digit numbers that you found. How do you know that you found them all?
Launching the task	• Warm-up problem • At the very beginning of the lesson:	• Warm-up problem • At the very beginning of the lesson:	• Warm-up problem • At the very beginning of the lesson:
Individual/	• Individual work	• Individual work	• Individual work

partner work			
Available resources	• Distribute a bag of Cuisenaire rods	• Not distribute a bag of Cuisenaire rods	• N/A
Issues that students have	• Difficulties with understanding what different solutions mean • Confusion with the Train Problem Part 1		• Confusion with writing number sentence for 10

8.2.1. Instructional Contexts

Instructional interactions are influenced by a number of factors of instructional contexts. Some factors exert more direct or greater influence on the dynamic of instructional interactions, but other factors exert somewhat indirect or minimal influence on the dynamic of instructional interactions. Among many other instructional contexts that might matter for instructional interactions, this section examines whether or not time point imposes different demands on constructing mathematical explanation. Another area of interest to examine would be the effect of class size on developing mathematical explanation or the comparison between developing mathematical explanation by heterogeneous group and developing mathematical explanation by homogeneous group but the EML data do not vary such instructional contexts.

Given that establishing norms at the beginning of an academic year has drawn much attention in the literature, it is worthwhile to examine whether or not students' development of mathematical explanation and the demands a teacher needs to accordingly deal with differ by the time points of when a mathematical task is taught. Because of the crucial role of establishing norms at the beginning of an academic year, the lessons delivered at the beginning of the academic year might sacrifice instructional time to the discussion of non-mathematical issues. One might consider that the different time point of teaching the same mathematical task dramatically changes what a teacher does or says to support students' development of mathematical explanation because instruction at the early time point of the program might require devoting a substantial amount of time to establishing norms such as enlisting students' participation, providing and hearing an explanation, making comments, and agreeing or disagreeing. In this sense, one might anticipate that teaching the same mathematical task at the early time point of the EML program might be less likely to engage students in mathematical practices, might be loaded with less cognitively demanding discourse moves (e.g., repeating rather than making comments), might sacrifice the intensity, depth, and quality of mathematical content to be discussed, and might take more time to get to the mathematical point than teaching the same mathematical task at the later point of the

EML program, thus significantly impacting on the quality of students' development of mathematical explanation. This raises the following questions:

1. Does the teacher's use of a particular discourse move for supporting students' development of mathematical explanation differ by the time point of teaching the same mathematical tasks?
2. Does the students' use of a particular discourse move for explaining their ideas differ by the time point of teaching the same mathematical tasks?
3. Does the quality of students' development of mathematical explanation, both individually and collectively, differ by the time point of teaching the same mathematical tasks?

In relating to the first and the second questions addressed above, it is evident in the data that the different time point of teaching the same mathematical task is not necessarily associated with the teacher's use of discourse moves and the students' use of discourse moves. No substantial differences are found in the teacher's use of discourse moves between when the same mathematical task is taught at the early point of the EML program and when the same mathematical task is taught at the later point of the EML program. After the teacher elicits an initial explanation for each proposed answer, she adopts various discourse moves such as (1) requesting either the explainer or the audience to repeat the initial explanation; (2) revoicing the initial explanation by herself; (3) asking for questions or comments; (4) asking for agreement or disagreement; (5) empowering the explainer to receive comments from the audience; and (6) using the shared established knowledge. The teacher's use of a particular discourse move is not particularly associated with when the same mathematical task is taught, but is more associated with whether or not a student's given explanation is hearable, understandable, and usable for other students; with whether or not the proposed answer is mathematically correct; and with the features of mathematical tasks. The students' use of a particular discourse move is not necessarily associated with when the same mathematical task is taught either, but it is more associated with the questions framed by the teacher.

For example, the brown rectangle problem has been introduced at the different time point of the EML program across five years. It is introduced at the early point in the EML 2008 (Day 1), the EML 2010 (Day 2), and the EML 2013 (Day 1), but introduced at the mid-point in the EML 2007 (Day 6) and the EML 2009 (Day 4). Because “agreement or disagreement” is often considered as a catalyst to enrich mathematical explanations, it is worthy examining how discourse moves have been used at the different time point of teaching the brown rectangle problem across five years.

Table 8.5 illustrates the number of times when the teacher or the students use “agree or disagree” for the brown rectangle problem. Here, I distinguish between the indirect use of “agree or disagree” and the direct use of “agree or disagree.” By the indirect use of “agree or disagree,” I mean the context in which the teacher mentions “agree or disagree” in a declarative sentence (e.g., “I think the question that probably there wouldn’t be a lot of disagreement about it, and then I’m gonna ask you a question about something that might be some more disagreement.”), whereas the direct use of “agree or disagree” refers to the context in which the teacher directly asks whether the students “agree or disagree” in an interrogative sentence (e.g., “Do you agree or disagree with that?”). In Table 8.5, the number indicates the instance of using “agree or disagree” rather than counting the exact number of each word. As shown in Table 8.5, the frequency of using “agree or disagree” by the teacher and the frequency of using “agree or disagree” by the students do not have a clear pattern associated with when the brown rectangle problem is taught. The direct use of “agree or disagree” neither randomly occurrs for any students’ responses nor mechanically apply for all students’ responses. Rather, the direct use of “agree or disagree” has been deliberately occurred for the following three cases: (1) checking for the agreement on the answer, both correct answer and incorrect answers for the first problem (equally partitioned rectangle); (2) seeking for the students who agree with the incorrect answer of 1/3 for the second problem (unequally partitioned rectangle); and (3) asking for the collective agreement on the established knowledge (whole; equal parts). It is interesting to see that the teacher asks “not to disagree” with the incorrect answers for the second problem.

Table 8.5. Frequency of using “agree or disagree” for the brown rectangle problem

	The teacher’s indirect use of “agree or disagree”	The teacher’s direct use of “agree or disagree”	The students’ use of “agree or disagree”
EML 2007 (Day 6)	5	0	2
EML 2008 (Day 1)	2	1	0
EML 2009 (Day 4)	3	8	3
EML 2010 (Day 2)	2	11	6
EML 2013 (Day 1, 2)	5	6	4

As another example, the blue and green rectangle problem has been introduced at the different time points of the EML program across five years. It is introduced at the early point in the EML 2008 (Day 3), the EML 2010 (Day 3), and the EML 2013 (Day 3), but introduced at the mid-point in the EML 2007 (Day 7) and the EML 2009 (Day 5). Table 8.6 illustrates the number of time when the teacher or the students use “agree or disagree” for the blue and green rectangle problem.

Similarly, the frequency of using “agree or disagree” by the teacher and the frequency of using “agree or disagree” by the students do not have a clear pattern associated with when the blue and green rectangle problem is taught. The higher frequency of direct use of “agree or disagree” in the EML 2009 and the EML 2010 could be explained by the fact that the EML 2009 cohort and the EML 2010 cohort have some practices with naming a fraction for more than one parts shaded using the working definition of fraction. The lower frequency of indirect use of “agree or disagree” for the blue and green rectangle problem than that of for the brown rectangle problem indicates that the need for legitimizing disagreement about the answers and for preserving the incorrect answers are relatively lower in the blue and green rectangle problem than in the brown rectangle problem.

Table 8.6. Frequency of using "agree or disagree" for the blue and green rectangle problem

	The teacher's indirect use of "agree or disagree"	The teacher's direct use of "agree or disagree"	The students' use of "agree or disagree"
EML 2007 (Day 7)	1	0	1
EML 2008 (Day 3, 4)	0	3	1
EML 2009 (Day 5, 6)	2	8	4
EML 2010 (Day 3)	0	7	0
EML 2013 (Day 3, 4)	0	3	2

In addition, for the same mathematical tasks, one might expect that the type of discourse move changes over time. For example, after an initial explanation elicited from students, a teacher might ask students to repeat the initial explanation at the early point of the EML program but ask for questions, comments, agreement, or disagreement at the later point of the EML program. However, even for the early point of the EML program, the teacher asks the audience to make comments or empowers the explainer to elicit comments from the audience. Again, the cognitive demand of discourse moves (e.g., repeating vs. commenting) does not have a clear pattern associated with the time point of when the mathematical task is taught.

Regarding to the third question, it is evident in the data that the different time point of teaching the same mathematical task does not impact the quality of students' development of mathematical explanation both individually and collectively. At the individual level, some students encounter the following problems with explaining: (1) declining the invitation to the board, refusing to explain in a public space, or appealing that they do not know how to explain; (2) having difficulties with speaking loudly enough and listening carefully to what others say; and (3) providing explanations or comments grounded on non-mathematical reasons. The small number of such cases makes it difficult to argue that these problems are particularly associated with the time point of when the mathematical task is taught. Even if such problems are encountered, resolving them in a mathematical way and providing sustained mathematical supports make the students to engage in the mathematical work.

At the collective level, regardless of when the mathematical task is taught, the students actively participate in mathematical practices such as trying to understand the reasoning of others' proposals, even if a proposal competes with or contradicts to their own proposals; being comfortable with expressing difficulties if they do not follow the flow of what is being explained or discussed; refuting others' reasoning with supportive data and constructing evidence to convince others; and making comments to the explainer despite supporting the same proposal. For example, the brown rectangle problem is taught at the very beginning of the EML program in 2008 and in 2010, but the EML 2008 students are actively engaged in making strong counterarguments to defend their answers and the EML 2010 students challenge the correct answer by asking "what if" questions to the initial explainer. To make this kind of mathematical work possible for the students, even at the very beginning of the EML program, the teacher makes the explanation provided by the students hearable, understandable, and available; distributes the turns equally to the students; builds responsibility for providing and hearing an explanation; and legitimizes disagreement about the answer.

The cross-year analysis of teaching the same mathematical tasks at the different time points reveals that it neither has a clear pattern associated with the quality of students' development of mathematical explanation nor imposes different demands to support students' development of mathematical explanation. Without sacrificing instructional time on discussing non-mathematical topics to establish norms, it is possible for the students to engage in the rich mathematical discussion from the very beginning of the academic year. However, it is important to notice that this intensive mathematical talk does not happen naturally, but is possible with the substantive mathematical supports deliberately provided by the teacher from the beginning of and throughout the program, such as serving as a delegate to address students' ideas in a public space; employing the prompts that build a mathematical correspondence; and making sure that the given explanation is hearable, understandable, and useable by requesting for repeating and asking for questions.

One caution for generalization would be that the EML is a two-week program, thus no substantial differences are found in the use of particular discourse moves across different time points. Over the academic year, it is possible that noticeable differences in the teacher's use of discourse moves and the students' use of discourse moves could be observed.

8.2.2. Pedagogical Approaches

Even for the instructions taught by the same teacher who has the same knowledge, skills, and disposition, the teacher has adopted different pedagogical strategies, techniques, and decisions. From the point of view that teaching is characterized as a set of routines, it might be somewhat unexpected to see such differences in pedagogical strategies, techniques, and decisions. As I discussed in the early section that the EML is a public teaching in which a number of pedagogical and mathematical ideas are tested by the EML observers' suggestion, it is difficult to conclude that these different approaches necessarily characterize the expert teacher's teaching practice. Instead, I examine how different pedagogical approaches might be interacted with the collective construction of mathematical explanation. In this section, I discuss (1) the selection of answers; (2) the sequence of answers; (3) the possible interaction between making public about what is monitored and constructing a mathematical explanation collectively; and (4) the possible interaction between organizing independent work and constructing a mathematical explanation collectively. In terms of the first two issues of selecting answers and sequencing answers, I mainly discuss the brown rectangle problem and the blue and green rectangle problem because they have a greater degree of variation of selecting and sequencing answers across years.

Selecting Answers

Selecting has been considered as one of the important pedagogical strategies to lead a whole-group discussion effectively (Stein et al., 2009). The selection could be done around representations or strategies, but this section mainly discusses about the selection of answers. A variety of selection methods could be considered. One way of selecting is to choose all the answers that the students produced in a private space. One merit of such method is the inclusiveness of all students' mathematical ideas in a public space, but this might be challenged by the limited time constraint. Another method is to choose key answers that are in accord with the targeted mathematical ideas to be discussed. Choosing the students who do not have enough chance to talk in a public space might be one alternative and choosing the students who produce the most complete, accurate, and clear written explanation in their notebooks might be another alternative. In selecting answers, it is not an easy task for a teacher to deal with if a student who produced the targeted answer in a private space does not want to share it in a public space or if the targeted answer is not brought up by the students in a private space or if a student who produced the least relevant answer vigorously expresses his or her willingness to present it in a public space. Given that the selected answers determine which mathematical ideas will be discussed, it is worthwhile to examine the following questions:

1. Are the selected answers for teaching the same mathematical tasks by the same teacher consistent across years?
2. How are the publicly aired answers selected? Are all of the answers that the students produced in a private space selected in a public space? Are the selected answers related to the proportion of answers that the students produced in a private space?
3. Are the monitored answers during independent work directly transferable to a public space?
4. Are there any answers that are deliberately aired or delayed for a further discussion by the teacher in a public space?

For the first question, it is evident in the data that the selected answers for teaching the same mathematical tasks by the same teacher are not consistent across years. Table 8.7 summarizes students' answers that are selected in a public space, students' answers that are not selected in a public space, answers that emerge from the students during instruction, and answers that are introduced by the teacher for the brown rectangle problem and for the blue and green rectangle problem. More specifically, in case of the brown rectangle problem, the incorrect answer of 1/2 for the first problem, which compares part-part rather than part-whole, was selected in the EML 2013 but not selected in the EML 2008 and EML 2010. The incorrect answer of 1/2 for the second problem, which takes a small part of the whole rectangle rather than the intended whole, was selected in the EML 2007 but not selected in the EML 2008. In case of the blue and green rectangle problem, the incorrect answer of 1/4, which takes a small part of the whole rectangle rather than the intended whole, was selected in the EML 2010 but not selected in the EML 2007, EML 2008, EML 2009, and EML 2013. The selected answers are not the same across years, but the teacher constantly makes space for the students to voluntarily propose in a public space. The teacher does not have a great control over which answers are selected but invites the students to propose answers in a public space, while introducing the key incorrect answers if necessary.

Table 8.7. Selecting answers in the brown rectangle problem and in the blue and green rectangle problem

Task	Year	Students' answers selected in a public space	Students' answers not selected in a public space	Answers emerged from the students	Answers introduced by the teacher
The brown rectangle problem: The first problem	EML 2007	1/3			
	EML 2008	1/3	1/2, 2/6, 3/9, 5/15		
	EML 2009	1/3		2/3, 2/6	
	EML 2010	1/3	1/2, 3/1		
	EML 2013	1/3, 1/2			
The brown rectangle problem: The second problem	EML 2007	1/4, 1/3, 1/2, $1\frac{1}{3}$			
	EML 2008	1/4, 1/3	1/2		
	EML 2009	1/4			1/3
	EML 2010	1/4, 1/3	4/1	1/6, 2/8, 4/16	
	EML 2013	1/4, 1/3, $1\frac{1}{2}$		Not a fraction	
The blue and green rectangle problem	EML 2007	1/2, 1/8	1/4 in half, $\frac{1/2}{4}$, 1/4, 4/8, $1\frac{1}{2}$, $3\frac{1}{2}$	1/5	
	EML 2008	1/2 of 1/4, 1/8, 1/2 + 1/4=1/8	1/2, 1/4, 2/4		1/2, 1/4
	EML 2009	1/8, 1/2	1/2 of 1/4, 1/4, 2/4, 8/4, 2/6		
	EML 2010	1/4, 1/8	1/2, 1/6, 1/5		
	EML 2013	1/8	1/4, 2/8, 1/2, 1/6, 8/8		

Regarding how the answers are selected, it is evident in the data that the publicly aired answers are selected by distributing equal opportunities for the students to provide an explanation in a public space rather than selecting the particular answers that the students produce. The teacher might be aware of what kind of answer that an individual student produce in a private space, but be more keen on giving an opportunity to students who did not have much opportunities to explain in a public space, regardless of the answers that he or she produces. In addition, not all of the answers that the students produced in a private space are selected in a public space. For example, in case of the brown rectangle problem, all of the answers that the students recorded in the notebooks are discussed in the EML 2007, but only some of the answers that the students recorded in the notebooks are discussed in other years. Given that the time constraint, one 90-minute lesson per each day for ten days, does not allow selecting all of the proposals in a public space, figuring out what needs to be placed in a public space is an important task of teaching.

In regard to the third question which addresses the transferability of monitored answers, what is monitored in a private space is not always directly transferable to a public space. The students' initial ideas are not static during the lesson, but evolve over time by hearing other students' mathematical ideas. In case of the brown rectangle problem, the EML 2009 students devise other answers such as 2/3 and 2/6 on the spot for the first problem and the EML 2010 students evolve their answers such as 1/6 or 2/8 or 4/16 for the second problem while hearing others' comments. In case of the blue and green rectangle problem, the students revise their answers while hearing others' explanations and make different proposals than what is recorded in their notebooks. Monitoring what kinds of mathematical ideas that students initially bring while they are working independently is important for supporting students to develop mathematical explanation, but keeping eyes on emerging mathematical ideas through the dynamic interactions with other students is the critical component of enriching the development of mathematical explanation.

Regarding whether answers are deliberately aired or further delayed in a public space, the key incorrect answers are introduced by the teacher if the students do not propose in a public space, while some answers are delayed for further discussion. For the brown rectangle problem, the incorrect answer of 1/3 for the second problem is introduced by the teacher to discuss the need for making equal parts. For the blue and green rectangle problem, the incorrect answer of 1/2 or the incorrect answer of 1/4 is introduced by the teacher to discuss the need for identifying the whole. On the other hand, for the blue and green rectangle problem, the discussion about algebraic transformation of "a fraction of a fraction" into a single fractional amount is delayed for further discussion by the teacher because the students do not equip the procedural knowledge about the multiplication of fraction. This illustrates that not all the incorrect answers have the equal mathematical status to be selected.

The cross-year analysis of selecting answers in teaching the same mathematical tasks by the same teacher reveals that the selected answers are largely influenced by the students' voluntary proposals rather than strictly being controlled by the teacher, while the key incorrect answers are introduced by the teacher if necessary. Monitoring what kinds of answers that the students produced in a private space is an important instructional resource for selecting answers, but it is also important to keep in mind that the students' mathematical ideas are not static throughout instructional interactions thus the new mathematical ideas may emerge. In addition to monitoring the kind of answers that the students originally produce in a private space, figuring out the proportion of particular answers, and determining mathematical value of the answers, it is also important to be open to eliciting different answers and to discuss the reasoning of the proposed answers in depth.

Sequencing Answers

Sequencing answers has been considered another important pedagogical strategy to lead a whole group discussion in a more mathematically coherent and predictable way (Stein et al., 2009). A variety of methods for sequencing could be considered. For example, Stein et al. (2009) illustrate several sequencing strategies: sequencing the strategies that a majority of students produce before the strategies that a few students produce, sequencing the strategies that are easy to understand before the strategies that are more complex, and sequencing the strategies that are based on the common misconceptions before the strategies that are mathematically correct ones. The sequencing might be done around representations or strategies, but this section mainly discusses the sequencing of answers.

One way of sequencing would be by the correctness of answers. One might imagine that sequencing the incorrect answer before the correct answer might create a much safer environment for the students who produce the incorrect answers because they are likely to be less intimidated by the accuracy, clarity, and completeness of explanation behind the correct answer. In addition, sequencing the incorrect answers before the correct answer might create a context that the incorrect answers are more likely to be proposed by the students in a public space. On the other hand, it can be imagined that sequencing the correct answer before the incorrect answers make enough space for discussing incorrect answers extensively. Another way would be to sequence the answers by the proportion of students who produce the answers, either from lower proportion to higher proportion or from higher proportion to lower proportion. Sequencing the answer that the majority of students produced before the answer that a few students produced makes enlisting students' active participation from the beginning, whereas sequencing the answer that a few students produced first before the answer that the majority of students produced is more likely to entice students to publicly air those answers. Even if the sequence of answers is consistently made by one of these approaches, the next important issue to be examined is whether there is an interaction between the correctness of answers and the proportion of answers in sequencing. Considering that the method of sequencing depends on the teacher's knowledge of students and particular instructional goals (Stein et al., 2009), it is worthwhile to examine the following questions:

1. Is the sequencing of answers for teaching the same mathematical tasks by the same teacher consistent across years? If so, what is the method of sequencing the answers? Is the sequencing determined by the correctness of answers? Is the sequencing determined by the proportion of answers that students produce in a private space? Is there interaction between the correctness of answers and the proportion of answers in sequencing?
2. Does the sequencing of answers impact on the dynamic of students' development of mathematical explanation? Does the first sequenced answer influence on the proposal of other answers or on the intensity of counterargument that can be made?

For the first question, it is evident in the data that the sequence of answers does not remain the same across years. In eliciting proposals, the teacher does not heavily control the sequence of the answers, but equally distributes opportunities for explaining to all students. Table 8.8 summarizes the sequence of answers in the brown rectangle problem and in the blue and green rectangle problem with information about whether the correct answer is more prevalent or not. In case of the brown rectangle problem, the correct answer was sequenced before the incorrect answers in the EML 2007, EML 2009, and EML 2010, but the incorrect answers were sequenced before the correct answer in the EML 2009 and EML 2013. When the correct answer is sequenced before the incorrect answers, the correct answer is not always more prevalent than the incorrect answers. In case of the blue and green rectangle problem, the correct answer was sequenced before the incorrect answers in the EML 2008, EML 2009, and EML 2013, but the incorrect answer was sequenced before the correct answers in the EML 2007 and EML 2010. The correct answer is more prevalent than the incorrect answers across years, but the sequencing does not remain the same across years.

Table 8.8. Sequencing of answers in the brown rectangle problem and the blue and green rectangle problem

Task	Year	Sequencing	Higher proportion of answers
The brown rectangle problem	EML 2007	Correct → Incorrect	Incorrect
	EML 2008	Incorrect → Correct	Incorrect
	EML 2009	Correct → Incorrect*	Correct
	EML 2010	Correct → Incorrect	Correct
	EML 2013	Incorrect → Correct	Similar
The blue and green rectangle problem	EML 2007	Incorrect → Correct	Similar
	EML 2008	Correct → Incorrect*	Correct
	EML 2009	Correct → Incorrect	Correct
	EML 2010	Incorrect → Correct	Correct
	EML 2013	Correct	Correct

Note: * indicates that the incorrect answers are introduced by the teacher.

For the second problem addressed above, the impact of sequencing answers on the dynamic of students' development of mathematical explanation varies by the mathematical tasks. In case of the brown rectangle problem, the way of sequencing answers does not quite impact the dynamic of developing a mathematical explanation.

One might imagine that sequencing a correct answer first and then an incorrect answer later might not be an effective way because the students who produce an incorrect answer might be intimidated by the completeness or rigorousness of mathematical reasoning of the correct answer thus they might be less willing to share their answers in a public space after hearing an explanation of the correct answer. Because of the important role of incorrect answers to develop mathematical explanation in the brown rectangle problem, one might imagine that sequencing the incorrect answer before the correct answer would be an effective way to support students' development of mathematical explanation. However, in the case of the brown rectangle problem, no evidence is found that some students who produce incorrect answers immediately change their positions after hearing the explanation of the correct answers. Rather, they tend to defend their positions or make counterarguments for the correct answer. In addition, by asking students not to agree or disagree on the proposed answer, the teacher might save enough space for discussing incorrect answers in a public space. In the brown rectangle problem, a pair of proposals is compared in a parallel structure rather than rigorously inspecting each proposal one by one. The richness of discussion to develop mathematical explanation for the brown rectangle problem is not affected by the sequence of answers.

In case of the blue and green rectangle problem, the way of sequencing answers seems to impact the dynamic of developing mathematical explanation. When the proportion of incorrect answers is relatively high, the incorrect answers are sequenced before the correct answer in the EML 2007 and EML 2010. On the other hand, when the proportion of incorrect answers is relatively low, the correct answer is sequenced before the incorrect answers in the EML 2008, EML 2009, and EML 2013. One reason might be that the chance to sequence the incorrect answer first is lower in those years because only one or two students produce those incorrect answers. One key issue to notice is that some students who produce incorrect answers change their answers after hearing the initial explanation of the correct answer. For example, in the EML 2007, after Mahluli's proposal of 1/2, Ethan proposed 1/5 and Daniel proposed 1/8. Both students, Ethan and Daniel, recorded incorrect answers by taking incorrect whole in their notebooks, but made different proposals from Mahluli, instead of supporting Mahluli's claim. One implication is that the relationship between correct answer and incorrect answers would influence how sequencing matters for developing mathematical explanation.

The cross-year analysis of sequencing answers in teaching the same mathematical tasks by the same teacher reveals that sequencing of the answers is not heavily controlled by the teacher. Both the correctness of answers and the proportion of the answers are not the main determinant of sequencing. The impact of sequencing answers on the dynamic of developing mathematical explanation seems to be different by the features of mathematical tasks. In some sense, both the terms "selecting" and "sequencing" seem to describe the way of controlling instruction by a teacher. However, it is likely that teachers do not have heavy control of selecting or sequencing, but need to encourage students to nominate different proposals and to give the equal opportunity to explain.

Making public what is monitored

Monitoring has been considered another important pedagogical strategy to orchestrate a whole-group discussion because it sets a basis for selecting and sequencing (Stein et al., 2009). On the one hand, what is monitored during independent work could be considered as a private resource only for the sake of a teacher; on the other hand, it could be thought about the ways in which what is monitored might be shared with students to facilitate the construction of mathematical explanation collectively in a certain way. One might imagine that sharing an observation about the answers only matters for the mathematical tasks with multiple answers, such as the two-coin problem and the three-permutation problem, but does not quite matter for the mathematical tasks with only one answer, such as the brown rectangle problem and the blue and green rectangle problem. Given that the way of sharing an observation about the answers by the same teacher does not remain the same in the data, it is worthwhile to examine the following questions:

1. Is there any interaction between sharing an observation about the answers and the publicly aired answers?
2. Is there any difference in sharing an observation about the answers across mathematical tasks?

Relating to the first question addressed above, sharing an observation about the answers might influence on the way of proposing multiple answers in a public space. In case of the brown rectangle problem, there are differences in sharing an observation about the answers that the students produce across years. The teacher shares the observation that the students came up with same answer for the first problem but they came up with different answers for the second problem in the EML 2007 and EML 2008, but the teacher does not share any observation about the answers the students produced either for the first problem or for the second problem in the EML 2009 and EML 2010. In the EML 2013, the teacher shares her observation that the students produced different answers for the brown rectangle problem, but does not specify which part of the problem. By the mathematical design of the brown rectangle problem, the incorrect answers, which violate the key ideas for naming a fraction (i.e., making equal parts; identifying the whole), are the core of developing mathematical explanation for the brown rectangle problem. It might be a coincidence, but when the teacher does not share the observation that the students produce the same answer for the problem, they propose additional answers for the first problem (2/3 by Sandra in the EML 2009, 2/6 by Callie in the EML 2009, removing the existing line by Jaclyn in the EML 2010, and 1/2 by Deshawn), despite the fact that the teacher asks for different answers after eliciting 1/3 for the first problem across all five years. These additionally proposed answers are not directly associated with the targeted key ideas for naming a fraction, so it does not directly contribute to the development of mathematical explanation but has a potential to indirectly contribute to the development of mathematical explanation. The proposal of 2/3 has a potential to develop the idea that the sum of all the parts equals to one whole and the proposal of 2/6 has a potential to develop the idea that adding an additional line makes an equivalent fraction. It is not just an issue about eliciting multiple answers, but it needs to consider how each proposed answer functions as a resource to develop a mathematical explanation. Depending on whether or not the teacher shares an observation about answers, the instructional time is managed differently. The teacher makes a quick transition from the first problem to the second problem in the EML 2007 and the EML 2008 (about one-minute whole-class discussion), with only one proposed answer for the first problem, but spends extended amount of time on the first problem with naming the unshaded parts and naming equivalent fractions in the EML 2009, with naming a fraction after removing the existing line in the EML 2010, and with naming a fraction incorrectly by focusing on a part-part relationship in the EML 2013, beyond eliciting a correct answer for the first problem. Sharing an observation about the same answer for the first problem allows the teacher to make a quick transition from the first problem to the second problem, but might limit the range of publicly aired answers.

For the second question which raises possible differences in sharing an observation about the answers across mathematical tasks, disagreement about the answers is often addressed for the brown rectangle problem but it is mostly not addressed for the blue and green rectangle problem. This could be explained by that the need for preserving incorrect answers is relatively higher for the brown rectangle problem than for the blue and green rectangle problem. For the two-coin problem and the three-permutation problem, even after monitoring students' work while they are working independently, the teacher does not elicit the number of answers that individual students produced in the notebooks. This could be explained by that explaining the exhaustiveness of answers is the key for the two-coin problem and the three-permutation problem.

The organization of independent work

The organization of group work has attracted much attention in the literature. An argument might be made around the effective organization of instruction for the entire instructional time (i.e., whole-group vs. group work vs. individual work), but this section limits the discussion about using individual work versus using partner work while students are working independently. Given that using a small-group has been considered as a promising method for learning mathematics, especially for 1980s and 1990s, one might imagine that using a partner work during independent work facilitates the development of mathematical explanation in a public space because sharing an explanation with a partner during independent work might reduce the anxiety of giving an explanation in a public space and create an opportunity of practicing a verbal explanation with a partner in a less intimidating environment; otherwise the students might just have an opportunity to produce written explanations or build their own internal language at most. Even if students are asked to produce written explanations during independent work, the successful transferability of written explanation to verbal explanation might not always be guaranteed. This section examines the following questions:

1. Is the organization of independent work by the same teacher consistent across years?
2. Does the use of partner work during independent work reduce the proportion of incorrect answers?
3. What would be the advantages of organizing partner work during independent work to construct mathematical explanation collectively during a whole-group? What would be the disadvantages of organizing partner work during independent work to construct mathematical explanation collectively during a whole-group?

For the first question addressed above, differences are found in setting up how the students work independently at the beginning of the lesson. In case of teaching the brown rectangle problem, the teacher makes a direct transition from individual work to whole-group discussion in the EML 2007, the EML 2008, the EML 2010, and the EML 2013, but asks the students to talk with a partner and to agree on the answer with a partner before launching a whole-group discussion in the EML 2009. In case of the blue and green rectangle problem, the students work individually in the EML 2007 but the students are assigned a partner to work with in the EML 2008 and the EML 2010. The teacher asks the students to work either individually or with a partner in the EML 2009 and the EML 2013. In case of the two-coin problem, the students work with a partner in the EML 2010 but the students work individually in the EML 2013.

Regarding to the relationship between the use of partner work during independent work and the proportion of incorrect answers, the proportion of incorrect answers is relatively lower when the students are assigned to work with a partner than when the students are working individually. In case of the brown rectangle problem, all but two students record the correct answer of 1/4 for the second problem in their notebooks in the EML 2009 when the teacher asks the students to talk with a partner and to agree on the answer during independent work, whereas less than half of the students record the correct answer of 1/4 for the second problem in their notebooks in other years except in the EML

2010. In case of the blue and green rectangle problem, the proportion of correct answer is the lowest in the EML 2007 when the students are working individually. In case of the two-coin problem, despite difference in the availability of coins to use, none of the students produce incorrect answers in the EML 2010 when the students are working with a partner, whereas a number of students produce incorrect answers in the EML 2013 when the students are working individually.

Lastly, despite the benefits of reducing the anxiety of giving an explanation and practicing a verbal explanation in a less intimidating environment, using a partner work during independent work raises another complex issue about how to engage both students in giving an explanation in a public space and how to distribute a different role of explaining to a pair of students. In addition, it is interesting to see that no incorrect answers for the second problem are proposed by the students in the EML 2009 when the teacher asks the students to agree on the answer with a partner during independent work. Obviously, engaging students in a partner work has benefits to share mathematical ideas and to practice with giving and hearing an explanation to each other, and to compare explanations in relatively safe environment of private space, but it also has the possibility that it does not preserve the key incorrect answers until a whole-group discussion.

8.2.3. Mathematical Approaches

The same mathematical tasks are taught by the same teacher, but several mathematical differences are observed across years. Among many other differences in teaching the same mathematical tasks such as the visual layout of the problem, the inclusion of written problem statement in the poster and in the handout, and the choice of language in the problem statement, this section examines differences in the use of concrete materials, the role of contextual features of mathematical tasks, and the role of instructional sequence of mathematical tasks. Given that using concrete materials and using real-life contexts have been recommended for teaching and learning mathematics, this section examines the possible interactions between such features and the development of mathematical explanations. In addition, by comparing instructional sequence, this section examines how the prior knowledge established from other mathematical tasks facilitates or impedes the construction of mathematical explanation for the given mathematical task. More specifically, this section examines the following questions:

1. Beyond the issues around the benefits of using concrete materials that are appropriate for the students' developmental stage, how does the use of concrete materials influence the construction of mathematical explanation?
2. Beyond the issues around the benefits of using real-life contexts that make the learning of mathematics meaningful and authentic for students, how do the contextual features of mathematical tasks influence the construction of mathematical explanation?
3. Does the instructional sequence of mathematical tasks influence the construction of mathematical explanation? If so, in what ways?

First, attending to mathematical affordances or limitations of using concrete materials is important for supporting students to develop mathematical explanation. Beyond the issues around the benefits of using concrete materials because they are appropriate for the students' developmental stage, an important issue to examine is how the use of such concrete materials plays a role in the construction of mathematical explanation. In case of the brown rectangle problem, the use of grid-poster allows proving the equal area from the measurement standpoint (counting the number of units in each part), whereas the use of the sticky brown rectangle allows proving the equal area from the geometry standpoint (translating the brown rectangle and showing the congruence of each part). In addition, the use of sticky line contributes to overcome the issue that adding a line changes the problem and resolving the issue of irreversible to the original problem; to see that drawing a line is a tool to make an easy access to see the equal parts rather than a determinant to name a fraction; and to think flexibly with adding an extra line or removing the existing line. In the blue and green rectangle problem, the use of sticky blue triangles and sticky green rectangles contributes to overcome the issue of dissecting the whole into different shapes by drawing lines and to increase the accessibility of direct comparison between the blue triangle and the green rectangle by geometrically transforming one shape into the other.

In case of the two-coin problem, the availability of coins influences the range of solutions that the students produced in a private space. When a bag of coins is distributed to the students in the EML 2010, none of the students produce the solutions which violate the conditions of the problem but a number of students produce the duplicated solutions. The use of coins prevents the production of incorrect solutions, but creates an issue of relying on the empirical trials of drawing coins to explain the exhaustiveness of multiple solutions rather than devising a method to systematically organize multiple solutions. The teacher prepares a bag of coins including sufficient and equal amount of coins deliberately planned, calculated, and controlled in advance rather than scooping a random number of coins on the spot. Without such careful planning, the impromptu careless use of coins could limit the production of all of the solutions because it is possible that the insufficient number of coins might be distributed to the students. On the other hand, when a bag of coins is not distributed to the students in the EML 2013, a number of the students produce the incorrect solutions which violate the conditions of the problem but few students produce the duplicated solutions. The unavailability of coins might alleviate concerns around the issue of relying on the empirical trials of drawing coins to explain the exhaustiveness of solutions, but a number of students struggle with finding the entering point of the problem that requires producing multiple correct solutions. In case of the three-permutation problem, the availability of Cuisenaire rods provides a visual clue for the students to find the structure of multiple solutions, but the same length of three Cuisenaire rods that are arranged in a different order might create a context for the students that the different arrangements look the same.

Second, the contextual features of mathematical tasks play a role in making the conditions of the problem, the intrinsic structure of finding multiple solutions, and the meaning of different solutions more explicit. In case of the three-permutation problem, three versions of permutation problems—which are mathematically isomorphic (Bass, in preparation)—are used with different contextual features: three-digit number, three-car train, and three-kids in a race. The three-digit problem is the most pure mathematical context, whereas the three-kids in a race problem is the most real-life context. Beyond the preference of using real-life contextual problems because of its benefits of authenticity and meaningfulness for students, it needs to examine how the contextual features of the three-permutation problem give an easy access or create difficulties to explain why the proposed solution is correct and how to prove the exhaustiveness of six solutions. More specifically, the three-digit problem creates an easy access to discern mathematically different solutions and to find the structure of six solutions by ordering the three-digit number from the smallest to the largest, whereas the three-car train problem creates difficulties to discern mathematically different solutions and to find the structure of solutions by ordering. Even in the mathematically isomorphic problems, the three-digit problem has an intrinsic structure of ordering the solutions whereas the three-car train problem does not have an intrinsic structure of ordering the solutions and the three-kid race problem needs for clarifying the possibility of ties of racing (Bass, in preparation). Depending on the contextual features of mathematical tasks, the probes are differently employed. In finding the structure of multiple solutions, the teacher asks "How are these two races different?" and "Which race is this one most like?" for the three-kid race problem, whereas she asks "What would you do next?" for the three-digit number problem.

Third, the instructional sequence of teaching mathematical tasks facilitates or impedes the construction of mathematical explanation. The instructional sequence of mathematical tasks formulates available mathematical knowledge, which is either validly or invalidly transferred, to construct mathematical explanation. I examine how the previously taught mathematical tasks, both relevant and irrelevant, increase an easy access or create difficulties to construct mathematical explanation for the given mathematical task. This could be examined by (1) transferring mathematical knowledge from relevant mathematical tasks and (2) transferring mathematical knowledge from irrelevant mathematical tasks.

The mathematical knowledge established from the brown rectangle problem would either facilitate or impede the construction of mathematical explanation for the blue and green rectangle problem. Given that both the brown rectangle problem and the blue and green rectangle problem reinforces the definitional ideas of naming a fraction, I examine how the knowledge established from the brown rectangle problem influences constructing mathematical explanation for the blue and green rectangle problem. Because the blue and green rectangle problem builds on the ideas established from the brown rectangle problem, the working ideas for naming a fraction and the evidences used to support the claim in the brown rectangle problem influences developing mathematical explanation for the blue and green rectangle problem. When the working definition of fraction is available to use in the EML 2007, EML 2009, EML 2010, and EML 2013, the initial student's explanation is followed by utilizing these ideas (identifying the whole and making equal parts). On the other hand, when the working definition of fraction is not explicitly established in the EML 2008, the initial student's explanation is followed by probing the reasoning behind the answer. As another example, in the EML 2008, counting the number of little squares in the grid is used to prove that the parts are equal in the brown rectangle problem, so the EML 2008 cohort often uses this prior knowledge to prove that the blue triangle and the green rectangle are the same size. In case of the blue triangle and the green rectangle, however, counting the little square is not an efficient method to prove equal size because of the shape of the blue triangle.

The mathematical knowledge established from the three-permutation problem would either facilitate or impede the construction of mathematical explanation for the two-coin problem, or vice versa. Because of its shared combinatory attributes but differences in taking an account of an order in arranging the objects, the instructional sequence of these two related mathematical tasks might have an influence on the production of answers and on the construction of mathematical explanation. Given that permutation (in which the order of arrangement matters) is mostly introduced before combination (in which the order of arrangement does not matter) in many mathematics textbooks, the different instructional sequences of these two related mathematical tasks in the EML provides an analytical opportunity to examine the effect of instructional sequence on producing solutions and on constructing mathematical explanation. More specifically, the two-coin problem is introduced before the three-permutation problem in the EML 2010, whereas the three-permutation problem is introduced before the two-coin problem in the EML 2013. Sequencing the three-permutation problem before the two-coin problem might lead to the production of 12 solutions for the two-coin problem whereas sequencing the two-coin problem first before the three-permutation problem might lead to the production of one solution for the three-permutation problem. Fischbein and Gazit (1988) found that the sixth graders and the eighth graders were better to solve the combination problem intuitively than the permutation problem before instruction but had more difficulties with the combination problem than the permutation problem after instruction because of the more complicated formula of combination than the formula of permutation.

In solving the three-permutation problem or the two-coin problem, the students make an invalid transfer of mathematical knowledge established from other irrelevant mathematical tasks. For example, in case of the two-coin, the EML 2013 students make an invalid transfer of mathematical knowledge established from writing a number sentence for 10 to writing possible combinations of two coins, which lead to the production of numerous incorrect solutions. As another example, in case of the three-permutation problem, the EML 2009 students make an invalid transfer of mathematical knowledge established from the Train Problem Part 1 to writing possible solutions of arranging three Cuisenaire rods, which leads to the confusion about mathematically different solutions.

In the cross-analysis of mathematical approaches across years, I neither make claims that a particular mathematical approach is better than the other in supporting students to develop mathematical explanation nor evaluates the teacher's knowledge to adopt a particular mathematical approach. Rather, this section aims to draw attention to the mathematical affordances and limitations of using concrete materials, the contextual features of mathematical tasks, and the instructional sequence of mathematical tasks to support students to develop mathematical explanation.

8.2.4. Students' Mathematical Ideas, Stances, Difficulties, and Issues

Given that one of the greatest predicaments of teaching is its dependence on students, it is important to figure out how instruction might unfold with different groups of students. On the one hand, one might imagine that instruction would unfold in the same way for teaching the same mathematical task by the same teacher because a teacher would make the same decisions based on his or her knowledge, skills, and disposition to achieve the same instructional goals. On the other hand, one might imagine that instruction would unfold in a dramatically different way even for the same teacher teaching the same mathematical task because teaching requires in-the-moment decisions to respond to the improvised and unexpected situations, mostly caused by the students. The question of how instruction unfolds with different groups of students might be answered based on one's personal sensibilities or perceptions built through years of their own teaching experiences, but it is not rigorously examined in the field yet how different groups of students are different and in what ways, whether or not such differences matter for the development of mathematical explanation, and what implications are for the work of supporting students to develop mathematical explanation.

Building on the individual-year analyses in the previous four chapters, this section examines (1) how instruction managed by the same teacher teaching the same mathematical task is likely to unfold differently with different groups of students (with likely areas of potential differences) and (2) how collective resources are likely to be differently constructed with different groups of students (because they are different students offering different opportunities). This section describes differences in mathematical ideas, stances, dispositions, and issues brought forward by each cohort and then discuss the possible contributions of collective resources that each cohort constructs for the development of mathematical explanation.

Considering that the EML students have sampled from the same school district over years—basically the same population with the similar ethnical, linguistic, racial, and socio-economical backgrounds—and the same procedure to recruit the EML students across years, it is presumed that there are no substantial differences in students' demographic backgrounds and their mathematical abilities across years. Despite the general homogenous features of the EML cohorts across five years, each cohort's mathematical ideas, stances, dispositions, and issues do not always remain the same. The observed differences are (1) the answers that the students individually produce in a private space and the answers that the students collectively discuss in a public space; (2) the proportion of the students who produce correct answer to the students who produce incorrect answers; (3) the intensity of counterarguments made against a competing proposal and the process of being convinced by a competing proposal; (4) when the key idea emerges; and (5) mathematical issues that matter the most for each cohort. This raises the following questions:

1. Does instruction unfold differently by different set of answers proposed in a public space? If so, to what extent does it change the dynamic of collective construction of mathematical explanation, mathematically and pedagogically?

Are the discourse moves customized or adjusted accordingly to the different set of answers proposed in a public space?

2. Does instruction unfold differently by different proportion of correct or incorrect answers? If so, to what extent does it change the dynamic of collective construction of mathematical explanation, mathematically and pedagogically? Are the discourse moves customized or adjusted accordingly to the different proportion of correct or incorrect answers?
3. Does instruction unfold differently by the intensity of counterarguments made and the process of being convinced? If so, in what extent does it change the dynamic of collective construction of mathematical explanation, mathematically and pedagogically?
4. Does instruction unfold differently by when the key idea emerges?
5. Does instruction unfold differently by different mathematical issues that matter the most for each cohort and the established knowledge by each cohort?

For the first problem, the answers produced individually in a private space and the answers discussed collectively in a public space are not the same across the cohorts. Given that eliciting multiple answers, solutions, and strategies has been considered an important aspect of teaching and learning mathematics, one important issue that needs to be examined is the relationship between the proposed answers and the development of mathematical explanation. In case of the first part of the brown rectangle problem (equally partitioned rectangle), only one correct answer (1/3) was proposed in the EML 2007, the EML 2008, and the EML 2010, but three answers (1/3, 2/3, and 2/6) were proposed in the EML 2009 and two answers (1/3, 1/2) were proposed in the EML 2013. After eliciting the correct answer of 1/3 and hearing an explanation about why it is 1/3, the teacher seeks different answers. This effort leads to the tacit collective agreement on the answer of 1/3 for the first problem in the EML 2007, the EML 2008, and the EML 2009, but leads to eliciting different answers in the EML 2009 and the EML 2013. These additionally proposed answers have the potential to be used to enrich mathematical explanation for the brown rectangle problem. The proposal of 2/3 for the first problem has a potential to develop the idea that the sum of the shaded part and the unshaded parts equals one whole. This can be used as collective resources to prove that the second problem cannot be called either 1/2 or 1/3, but extra mathematical work is needed for students to use this collective resource. The proposal of 2/6 for the first problem has a potential to develop the idea that adding additional lines produces equivalent fractions. This can be used as collective resources to support the idea that drawing an additional line makes four equal parts for the second problem, but the premise of preserving equal parts needs to be more explicitly addressed. The proposal of 1/2 for the first problem is important resource to make explicit the idea that the area model of fraction is not a part-part relationship (the number of shaded parts vs. the number of unshaded parts) but a part-whole relationship (the number of shaded parts vs. the total number of equal parts), thus remediating the invalid reasoning about naming a fraction, but it does not directly contribute to develop the key ideas of naming a fraction—either making equal parts or identifying the whole. The EML 2009 cohort and the EML 2013 cohort create additional collective resources in discussing the first problem, but are neither used to further develop the targeted mathematical ideas (i.e., "making equal parts" and "identifying the whole") nor transfered to explaining the second problem.

For the second part of the brown rectangle problem (unequally partitioned rectangle), four answers (1/4, 1/3, 1 and 1/3, 1/2) were proposed in the EML 2007, two answers (1/4 and 1/3) were proposed in the EML 2008, one answer (1/4) was proposed in the EML 2009, five answers (1/4, 1/3, 1/6, 2/8, 4/16) were proposed in the EML 2010, and three answers (not a fraction, 1/4, and 1 and 1/2) were proposed in the EML 2013. When the key incorrect answer (1/3) was not proposed by the students, the teacher introduced the incorrect answer and asks them the reasons for not calling it as 1/3 in the EML 2010. In the EML 2013, the key incorrect answer (1/3) was not directly proposed by the students, but the idea of "not equal" was offered by Kadeem and 1/3 was indirectly refuted by Kallie's explanation for her proposal of 1 and 1/2. Unlike the first problem where the teacher directly asks for agreement or disagreement with the proposed answers, the teacher does not ask for agreement or disagreement with the proposed answers for the second problem. This could be explained by the fact that the teacher makes efforts to preserve the incorrect answers for the second problem to develop mathematical explanation. The incorrect answer of 1/3 contributes to develop the idea of "equal parts" and the incorrect answer of 1/2 contributes to develop the idea of "identifying the whole."

Depending on whether 1/2 is proposed or not, the knowledge established for naming a fraction at the end of lesson is different. The idea of "making equal parts" is established for all years, but the idea of "identifying the whole" is established only for the years when either 1/2 (in the EML 2007) or 1 and 1/2 (in the EML 2013) is proposed. For other years that the incorrect answer of 1/2 is not proposed in the EML 2008, EML 2009, and EML 2010, the key idea of "identifying the whole" is developed in discussing the blue and green rectangle problem.

In addition, depending on the proposed answers for the second problem, the teacher customizes her questions differently. Table 8.9 illustrates how the questions are customized by the type of proposals made in a public space. In the EML 2007 when four answers were proposed, the teacher takes a neutral stance to compare the difference between 1/3 and 1/4 without narrowing the scope of candidate answers into these two answers. In the EML 2008 in which only two competing proposals were made, the teacher takes a stronger stance by using a modal verb of "should" to make a choice between 1/3 and 1/4. In the EML 2009 when no one proposed 1/3, the teacher takes a stance of persuading students to seriously consider 1/3. In the EML 2013, the teacher takes a stance that both 1/4 or not a fraction could be options.

Table 8.9. Customized questions by the type of answers proposed in a public space

	Proposed answers	The teacher's questions
EML 2007	1/4; 1/3; 1/2; 1 and 1/3	"What's the difference between the one-third and the one-fourth?"
EML 2008	1/4; 1/3	"Does anybody think they have a reason why it ***should be*** one of these or the other?"
EML 2009	1/4	"Well, ***isn't that right*** because it's one out of three then. So ***wouldn't we call*** that one-third?"
EML 2010	1/4; 1/3	"So ***is*** this one right here, one-third or one-fourth."
EML 2013	not a fraction; 1/4, 1 and 1/2	"We have one-fourth ***might*** be an answer or not a fraction."

In case of the blue and green rectangle problem, only one correct answer (1/8) was proposed by students in the EML 2008 and EML 2013, but other incorrect answers (1/2 or 1/4) were proposed by students in the EML 2007, EML 2009, and EML 2010.

Even in the context that only one correct answer (1/8) was proposed, the EML 2008 cohort deals with different representations, such as 1/2 of 1/4 or 1/2 + 1/4. In the context which the cohort does not establish the knowledge that identifying the whole is an important idea for naming a fraction from the brown rectangle problem, incorrect answers (1/2 and 1/4) are introduced by the teacher to develop the key definitional idea of identifying the whole in the EML 2008.

Even though a teacher makes a similar attempt to elicit multiple answers, different groups of students are likely to bring a different set and number of answers in a public space. Beyond attending to the number of answers produced in a public space, an important task of teaching includes (1) not dismissing any proposals made in a public space but unpacking the reasoning behind the proposals; (2) introducing the key incorrect answers if they are not brought by students; (3) mapping the proposed answers onto the targeted mathematical idea to develop; (4) deciding what needs an immediate agreement or disagreement and what needs to be preserved; and (5) customizing questions and prompts responding to the relationship between proposed answers (whether the proposed answers are mutually exclusive or compatible).

For the second question, the proportion of the students who produce correct answer to the students who produce incorrect answer is not the same across cohorts. Nearly all of the students come up with the correct answer for the first part of the brown rectangle problem across five years, but for the second part of the brown rectangle problem, the incorrect answers are more prevalent than the correct answer in the EML 2007, the EML 2008, the EML 2010, and the EML 2013, but most of the students recorded the correct answer in the EML 2009. In case of the blue and green rectangle problem, the correct answer is more prevalent than the incorrect answers across years, but the proportion of incorrect answers produced by not taking the intended whole is relatively higher (greater than 32%) in the EML 2007 and EML 2010, whereas the proportion of incorrect answers produced by not taking the intended whole is pretty low (less than 7%) in the EML 2008 and EML 2013. A number of students completely crossed out their original answers in their notebooks, so it is difficult to make a strong argument for how the exact proportion of the students who produce correct answer to the students who produce incorrect answer (a case which most students get the correct answer; a case which most students get the incorrect answer; and a case which the half of the students get the correct answer but the other half get the incorrect answer) influence the dynamic of developing mathematical explanation, but there is a possibility that the proportion of correct answer to the incorrect answer might be related to the mathematical stance that the students bring to the instruction. If the proportion of the students who produce correct answer to the students who produce incorrect answer matters in developing mathematical explanation, an important task of teaching includes (1) surveying the composition of students' mathematical ideas either by assigning it as a homework in advance, collecting the data while circulating the classroom, or asking students to raise their hands; (2) protecting that the mathematical stance is not influenced by the idea that a majority of students or the advanced students have; and (3) customizing questions and prompts by the proportion of the students who produce correct answers to the students who produce incorrect answers.

For the third question, the intensity of counterarguments made against a competing proposal and the process of being convinced by a competing proposal are not the same. For example, in case of the brown rectangle problem, in the EML 2007, the incorrect answer of 1/3 for the second problem is first proposed by Stan, refuted by Lila in explaining her answer of 1/4, explained by Roddie, repeated by Micah and Mahluli, but no further defense was made by either Stan, Roddie, or other students. After hearing why the second problem tricks people by not having a line from Christopher and Marcel and why it is important to draw a line by Daniel and Amber, the EML 2007 cohort is quite convinced by why it cannot be called 1/3. In the EML 2008, the incorrect answer of 1/3 for the second problem is proposed by Chantal and further defended by Karl, Calder, and Manoel by arguing that the second problem comes out without a line so drawing a line changes the problem. The counterarguments were strongly made even after Britney brought the idea that fraction needs to be divided into equal parts, but reconciled after Alexico's examples to prove that the line does not matter. In the EML 2009, the incorrect answer of 1/3 for the second problem is introduced by the teacher, but no other students support this claim. Malik explained why someone might call 1/3, but Teri, Mannis, and Marlais brought the idea that the parts need to be equal. In the EML 2010, the incorrect answer of 1/3 for the second problem is proposed by Macaulay and defended by himself, but no one provides further defense. After hearing that 1/3 does not consider equal parts from Jaclyn and Eric, Macaulay changed his mind. In the EML 2013, the incorrect answer of 1/3 for the second problem is indirectly refuted by Kallie in proposing her answer of 1 and 1/2. Like this, some cohorts are more easily convinced by the idea that adding a line makes equal parts because the problem tricks people by not having a line (e.g., EML 2007), but others are more resistant and hesitant to accept the idea because it contradicts their general perception that adding a line changes the problem (e.g., EML 2008). The process of reconciling the competing proposals is not the same across five cohorts, but all of the cohorts ultimately arrive on the agreement that making equal parts is an important idea for naming a fraction and drawing a line provides an easy access to see equal parts. As the intensity of counterargument and the resistance of accepting the competing proposal gets higher, students construct more rich collective resources to convince others who have a competing proposal. It is not an easy task for a teacher to have students take a strong stance on their mathematical ideas and to have them sustain their perseverance until they are fully convinced, but detecting such a moment, confronting competing ideas, and providing sufficient opportunities to defend one's proposal is an important task for the work of supporting students to develop mathematical explanation.

For the fourth question, the key idea emerges at the different stage of developing the mathematical explanation. In case of the brown rectangle problem, the key idea of "equal" is early proffered by Lila in proposing her answer of 1/4 in the EML 2007 and by Kadeem in proposing his answer of "not a fraction" in the EML 2013. Lila's idea of "not equal" is not immediately taken up by the teacher in the EML 2007, but Kadeem's idea of "not equal" is immediately used by Liberty in supporting her answer of 1/4 and by Kallie in refuting the answer of 1/3 indirectly. On the other hand, the key idea of "equal" emerges in the process of comparing between the equally partitioned rectangle and the unequally partitioned rectangle in the EML 2008 (by Britney and Melody), EML 2009 (by Teri and Marlais), and EML 2010 (by Dahlia). Even after a cohort comes up with the idea of "equal," searching for an appropriate vocabulary to define the meaning of "equal" takes a different route. Amber comes up with "equal parts" in the EML 2007, Britney suggests " equal parts" and Melody elaborates it into "the same amount of space" in the

EML 2008, Teri points out “different shapes” but Marlais elaborates it into “different sizes” in the EML 2009, Dahlia come up with “equal parts” in the EML 2010, and Kadeem gets “equal shape” and then elaborates it into “equal parts” in the EML 2013. The EML 2009 cohort reaches an early agreement on the answer of 1/4 for the second problem, but takes time to search for the appropriate vocabulary and to pay attention to same area rather than same shapes (because different shapes could have the same area as shown in the blue and green rectangle problem). Deriving the targeted mathematical idea and the accurate mathematical language is key for developing mathematical explanation. One important task of teaching is not just satisfying with the quick offering of the targeted mathematical idea and using the accurate mathematical language by an individual student, but providing sufficient supports for students to use those collective resources.

For the last question above, the mathematical issues that matter the most for each cohort are not always the same. In case of the brown rectangle problem, the EML 2007 cohort spent a significant amount of time to make sense of 1/2 for the second part of the problem, the EML 2008 cohort discussed whether or not the line is allowed to draw for the second part of the problem, the EML 2009 cohort discussed naming unshaded parts and naming an equivalent fraction for the first part of the problem, the EML 2010 cohort engaged in removing the existing line or adding an additional line to make unequal parts, and the EML 2013 cohort spent time to make sense of 1 and 1/2 for the second part of the problem. In case of the blue and green rectangle problem, the EML 2007 cohort dissected the whole into the same shapes (eight triangles for naming the blue triangle and eight rectangles for naming the green rectangle), but the EML 2008 cohort, EML 2010 cohort, and EML 2013 cohort did not dissect the whole into the same shapes (e.g., four triangles and four rectangles for naming the blue triangle; six triangles and two rectangles for naming the blue triangle; six rectangles and two triangles for naming the green rectangle). The issue of dissecting the big rectangle into different shapes is raised by the EML 2010 cohort and by the EML 2013 cohort, but it is not raised by the EML 2008 cohort. In case of the two-coin problem, the EML 2010 students have an issue of repeating the same solutions, whereas the EML 2013 students have an issue of producing the incorrect answers which violate the conditions of the problem. Depending on the mathematical issues that each cohort struggles, adjusting instructional time and using resources accordingly is an important task of teaching.

In basic ways, the students and their mathematical proficiency are similar across years, but each cohort brings somewhat different mathematical ideas, stances, dispositions, difficulties, and issues even for the same mathematical tasks. Thus each cohort develops somewhat different collective resources that become available for use either by a teacher or by students. In comparing the mathematical ideas, stances, dispositions, difficulties, and issues brought by different groups of students, I offer the following observations. First, the mathematical scope and terrain of collective resources that each cohort establishes varies to a certain degree, but all of the cohorts develop the key ideas for developing mathematical explanation. Second, there are variations in what collective resources are available for use to develop a mathematical explanation across years, but the practice of constructing collective resources is quite the same. Third, some collective resources are immediate or necessary use, but others remain in reservoir or optional for use either by a teacher or students. Fourth, the same mathematical issue is treated differently because of the cohort's established knowledge and the students' mathematical stance. For example, in case of the brown rectangle problem, both the EML 2007 cohort and the EML 2008 cohort address the issue that the line does not matter, but the teacher redirects the discussion on why it is important to draw a line in the EML 2007, whereas the teacher let the students to engage in whether or not the line matters in the EML 2008. At the time point when this issue is addressed, the EML 2007 cohort does not establish the idea that drawing a line make equal parts and does not make a strong argument about 1/3, whereas the EML 2008 cohort already established the idea that drawing a line makes equal parts and has a strong argument about 1/3. Lastly, eliciting multiple answers has been considered important pedagogical practice for fostering students' mathematical abilities and enriching mathematical discussion, but it needs to be examined how the proposed answer could be used as resource for enriching the development of mathematical explanation.

8.3. Supporting Students To Develop Mathematical Explanation: Findings From the Cross-Mathematical-Task Analysis

Given that the ideas around leading a whole-group discussion and creating a mathematical discourse community are more generic but are not much differentiated by mathematical tasks in the literature, the question remains whether supporting students to develop mathematical explanation might be dependent on mathematical task, and if so, in what ways. In a broader sense, Leinhardt (2001) addresses the issue of "subject-matter generality" and "subject-matter specificity" in conceptualizing instructional explanations. She writes:

> So far, the description of instructional explanations has floated relatively free from any subject matter anchor; however, what prompts a query for an explanation is something embedded in the subject matter itself. […] In describing these differences in the intellectual moments for instructional explanation, we must think in terms of the kinds of epistemic structures that organize a domain. We must also think of the kinds of social and psychological features that challenge the student while learning a domain. (Leinhardt, 2001, p.342)

In a similar vein, different branches of mathematics (e.g., algebra, geometry, and probability) have their own epistemology, logic, and language. In this sense, it is worthwhile to examine "mathematical-task generality" and "mathematical-task specificity" in conceptualizing the work of teaching entailed in supporting students to develop mathematical explanation.

In constructing mathematical explanation, there are some general issues of explaining in the individual level such as vagueness, lack of clarity, stammering, redundancy, and repetition of explanation and some general issues of explaining in the collective level regarding to hearing others' explanations in an attentive way and expressing disagreement with others' explanations in a respectful way. To resolve those issues, a number of general pedagogical moves, strategies, and techniques are commonly employed regardless of the types of mathematical tasks, such as enlisting students' participation, making the explanation hearable to the audience, asking for repeating or revoicing, asking for questions or comments, and asking for agreement or disagreement.

Acknowledging "mathematical-task generality" in supporting students to develop mathematical explanation, however, it is likely that mathematical reasoning, warrant, logic, and structure required to explain one mathematical task might be different from mathematical reasoning, warrant, logic, and structure required to explain another mathematical task. Supposing that the specialized form of mathematical reasoning is required for explaining each mathematical task, the mathematical issues that students struggle with explaining might be different across different mathematical tasks. If this is the case, mathematical supports that teachers need to provide would differ accordingly. This section aims to examine the features of "mathematical-task specificity" in supporting students to develop mathematical explanations.

The next section begins by providing a brief summary of building mathematical explanation in the brown rectangle problem, the blue and green rectangle problem, the two-coin problem, and the three-permutation problem. Following this summary, I refine the model of mathematical-task specificity of constructing mathematical explanation by laying out the relationship between the nature of mathematical explanation entailed in mathematical tasks, the struggles faced by students, and the mathematical supports provided by the teacher for the aforementioned four mathematical tasks. In the last section, I discuss the need for developing the categorization of mathematical tasks aligned with the nature of mathematical explanation.

8.3.1. A brief summary of building mathematical explanation in the four mathematical tasks

The key elements of mathematical explanation for the brown rectangle problem are developing the concepts for a definition of a fraction—making equal parts and identifying the whole. The idea of "making equal parts" is mainly developed by contrasting between the correct answer (1/4) which takes equal partitioning into account and the incorrect answer (1/3) which does not take equal partitioning into account. In a similar vein, the idea of "identifying the whole" is mainly developed by contrasting between the correct answer (1/4) which accurately identifies the intended whole and the incorrect answer (1/2) which misidentifies the intended whole. Due to the important role of incorrect answers in developing a definition of fraction, the efforts to preserve the key incorrect answers are made by the teacher, such as not remediating the key incorrect answers during independent work and legitimizing the key incorrect answers at the beginning of whole-group discussion. In the brown rectangle problem, the examples employed by the students strengthen the arguments, such as Alexico's example of deleting the existing line for the second part of the brown rectangle problem and shifting the shaded part for the second part of the brown rectangle problem in the EML 2008 or Jaclyn's example of deleting the existing line for the first part of the brown rectangle problem and Coretta's example of adding two additional lines for the second part of the brown rectangle problem in the EML 2010. Despite the differences in whether or not maintaining the unequally partitioned status, the students employed examples in ways which skillfully manipulate the key mathematical ideas to support their claims.

In addition to developing the concepts for a definition of a fraction, the blue and green rectangle problem aims to elaborate the concept of equal by proving the equality of two different shapes (the blue triangle vs. the green rectangle). Sharing the similar nature of mathematical explanation with the brown rectangle problem, however, the availability of the working definition of fraction established from the brown rectangle problem influences the kinds and the level of instructional supports provided by the teacher for the blue and green rectangle problem. As shown in the previous chapters, after eliciting an initial explanation, the teacher asks the students for repeating, revoicing, questioning, and commenting without readily filling the missing information in the initial explanation offered by the students for the brown rectangle problem, but she asks the students to utilize the working definition of fraction and sometimes fills the missing information in the initial explanation offered by the students for the blue and green rectangle problem. Unlike the brown rectangle problem, examples are not employed by the students to strengthen their arguments. In both mathematical tasks, due to difficulties of describing different parts of the pictorial representation distinctively and articulately, the teacher invites the students to the board to supplement the often-vague verbal description and to build a correspondence between a verbal explanation and a pictorial representation. For both mathematical tasks, explanation is required to justify the answer and justifications lead to the formulation of the part-whole definition of a fraction.

On the other hand, the key elements of mathematical explanation for the two-coin problem and the three-permutation problem are (1) justifying individual solutions that meet the conditions of the problem and (2) justifying the exhaustiveness of multiple solutions by systematically organizing all of the solutions. In spite of the important mathematical difference between the two-coin problem (i.e., the order of arrangement does not matter) and the three-permutation problem (i.e., the order of arrangement matters), they share the similar nature of mathematical explanation in a number of ways. Unlike the efforts to preserve the key incorrect answers in the brown rectangle problem and the blue and green rectangle problem, the efforts to eliminate the incorrect answers are made by the teacher at the beginning of the lesson. Being unable to eliminate those incorrect answers at the beginning of the lesson makes it difficult for the students to engage in the productive discussion about the exhaustiveness of multiple solutions. The process of eliminating the incorrect answers at the beginning of the lesson contributes to clarify the conditions of the problem. A public discussion about incorrect answers is beneficial to clarify the conditions of the problem but it is not always necessary for the two-coin problem and for the three-permutation problem if students do not have struggles with it.

Due to the importance of systematic organization of solutions, the teacher has a heavy control of using a public space to coherently and sustainably keep the records of the students' work on the board rather than having the students record solutions randomly and temporarily in a public space for the two-coin problem and for the three-permutation problem. Additionally, unlike the attributes of employed examples for the brown rectangle problem, the example employed by the student was not effective to strengthen the argument for the three-permutation problem. For example, in making a generalization that there are six answers for arranging three objects with a different order in the EML 2010, Thailee employed an example which simplifies the original permutation problem. Thailee claimed that there are four solutions for arranging two objects with a different order but the claim is mathematically incorrect one by applying the multiplicative reasoning rather than combinatorial reasoning.

Beyond the general issues of providing, hearing, constructing, and challenging an explanation which are commonly observed regardless of the type of mathematical tasks, the students face mathematical-task specific struggles in developing mathematical explanation. In explaining the brown rectangle problem and the blue and green rectangle problem, the students had issues with inaccurate, incorrect, incoherent, demonstrative, transitional, and mathematically ungrammatical language use; vague referent to the pictorial representation; and unfulfilled logical structure. In addition, the rigorous criteria for adopting accurate terms would be challenging for the students to develop mathematical explanation. The use of different "shapes" in refuting the incorrect answer of 1/3 for the second part of the brown rectangle problem might be sufficient for the brown rectangle problem, but it leads to the invalid conclusion for the blue and green rectangle problem in that they have different shapes but the same area. In addition, the term of "shape" is specific to the area model of fraction but not generalizable to name a fraction for the set model or the number line model.

On the other hand, in explaining the two-coin problem and the three-permutation problem, the students had issues with relying on the empirical trials and invalid transfer of prior knowledge (e.g., applying multiplicative reasoning for the context which requires combinatorial reasoning) in developing mathematical explanation. The students often use the informal language of "switching around" or "flipping around" to explain changing the order of arrangements, but the need for scaling up those words to the formal mathematical term is not high. Thus, the rigor of accurate language use is different across mathematical tasks.

Table 8.10 summarizes the key elements of mathematical explanation, the role of incorrect answers, the struggles faced by the students, the affordances offered by the collective work, the mathematical supports provided by the teacher during independent work, the use of a public space, and the use of discourse moves for teaching the four mathematical tasks.

Table 8.10. A comparison of instructional features across four mathematical tasks

	MATHEMATICAL TASK 1: THE BROWN RECTANGLE PROBLEM	MATHEMATICAL TASK 2: THE BLUE AND GREEN RECTANGLE PROBLEM	MATHEMATICAL TASK 3: TWO-COIN PROBLEM	MATHEMATICAL TASK 4: THREE-PERMUTATION PROBLEM
Key elements of mathematical explanation	• Developing the concepts for the working definition of a fraction o Making equal parts o Identifying the whole (depending on the proposed incorrect answer) • Building a connection between a verbal explanation and a pictorial representation; between an answer and an explanation • Using language accurately, correctly, precisely and consistently	• Utilizing the working definition of a fraction o Making equal parts o Identifying the whole • Building a connection between a verbal explanation and a pictorial representation; between an answer and an explanation • Using language accurately, correctly, precisely and consistently • Extending the meaning of "equal" from "equal shapes" to "equal size (area)" • Proving the congruence of different shapes	• Explaining why each proposal is acceptable as a correct answer by using the conditions of the problem • Understanding the meaning of "at most twice" (even though it is implicitly stated in the problem) • Justifying that the order does not matter • Understanding what mathematically different solutions mean • Finding the mathematical structure rather than grounding on the empirical reasons	• Explaining why each proposal is acceptable as a correct answer by using the conditions of the problem • Understanding the meaning of "exactly once" (it is explicitly stated in the problem) • Justifying that the order matters • Understanding what mathematically different solutions mean • Finding the mathematical structure rather than grounding on the empirical reasons
The role of key incorrect answers	• The relationship between correct answer (1/4) & incorrect answer (1/3) is mutually exclusive (contradictory). • The incorrect answer of 1/3 is caused by lack of understanding about the key definitional idea • Preserved until the later point • Introduced, if it is not proposed by students • No remediation of the key incorrect answers during individual work • Making the key definitional idea explicit: making equal parts	• The relationship between correct answer (1/8) & incorrect answer (1/2 or 1/4) is inclusive. • The incorrect answer of 1/2 or 1/4 is caused by lack of clarification about the problem statement (intended vagueness) • Introduced if necessary, but not always • Remediation of some incorrect answers (e.g., 1/6 or 2/6 or 2/8) or extending to the intended whole • Making the key definitional idea explicit: identifying the whole	• Incorrect answers violate the conditions of the problem. • Eliminated the incorrect answers at the beginning • Remediation of errors during individual work • Making the conditions of the problem explicit	• Incorrect answers violate the conditions of the problem. • Eliminated the incorrect answers at the beginning • Remediation of errors during individual work • Making the conditions of the problem explicit
Students' difficulties	• Do not equip with the key concept or do not make it explicitly • Using inaccurate, incorrect,	• Do not isolate irrelevant information (e.g., counting the blue triangle and the green rectangle together; drawing a line	• Duplicated solutions (especially, when using coins) • Producing solutions which violate the conditions of the problem	• Lack of understanding about the key terms (e.g., three-digit number vs. digit three) • Producing solutions which violate

	inconsistent, imprecise, and vague language to describe the object • Paying attention to the partial components of naming a fraction • Using pre-defined terms (e.g., numerator or denominator) • Prior misconception (e.g., drawing a line changes the problem) • Do not know what to explain or how to explain	on the blue triangle influences on naming a fraction for the green rectangle) • Lack of clarification about the term (i.e., “big rectangle”) in the problem statement • Using inaccurate, incorrect, inconsistent, imprecise, and vague language to describe the object • Do not have the established knowledge to handle “a fraction of a fraction” algebraically • Prior misconception (e.g., can’t make equal with different shape)	(especially, when not using coins) • Not getting to the entry point of the mathematical task because of the lack of understanding about the problem • Do not produce enough solutions • Not having a systematic organization • Being confused with other mathematical tasks (e.g., writing number sentence) • Do not establish what counts as an acceptable mathematical explanation	the conditions of the problem • Not getting to the entry point of the mathematical task because of the lack of understanding about the problem • Do not produce enough solutions • Not having a systematic organization • Being confused with other mathematical tasks (e.g., Train problem part 1) • Do not establish what counts as an acceptable mathematical explanation
Collective work	• Making counterargument with the supports of examples and non-examples • Appropriating the accurate terms • Bringing the incorrect answer	• Refining/revising the answers after hearing others’ explanation	• Increasing the accessibility to all solutions • Finding a structure of solutions and applying the reasoning to reorganize solutions	• Increasing the accessibility to all solutions • Finding a structure of solutions and applying the reasoning to reorganize solutions
While launching	• No further or extensive clarification about the problem	• No further or extensive clarification about the problem	• Reading aloud the problem statement • Restating what the problem is asking • Eliciting examples or non-examples to clarify the conditions of the problem	• Reading aloud the problem statement • Restating what the problem is asking • Eliciting examples or non-examples to clarify the conditions of the problem
Circulating during individual or partner work	• Check students’ work but do not provide substantive mathematical support spontaneously • Do not remediate the key incorrect answers • Resolving the issues in a private space upon request, but not addressing it in a public space	• Clarifying the whole upon the individual student’s request in a private space • Remediating some errors (e.g., not making equal parts; naming the blue triangle and the green rectangle together) that impede the accessibility to the key ideas • Extending the partial whole into the intended whole (optional) • Resolving the issues in a private space upon request, but not addressing it in a public space	• Remediating the key incorrect answers • Addressing the issues raised by individual request in a public space • Spotting the resources that boost up a whole-group discussion • Bringing the individual issues/struggles in a public space	• Remediating the key incorrect answers • Addressing the issues raised by individual request in a public space • Spotting the resources that boost up a whole-group discussion • Bringing the individual issues/struggles in a public space

Use of a public space	• Invite students to the board (to point out the pieces)	• Invites students to the board (to point out the pieces)	• Controlled by the teacher for the systematic organization	• Controlled by the teacher for the systematic organization
Discourse moves	• Neither agree nor disagree with the key incorrect answers • More request of repeating or revoicing after eliciting an initial explanation at the beginning	• Neither agree nor disagree with the key incorrect answers • Less request of repeating or revoicing after eliciting an initial explanation but making a reference to the working definition of fraction • Correct terms and filling the missing information in a natural way but not in a distracted way	• Less request of repeating or revoicing after eliciting each proposal • Asking for the collective agreement about each proposed answer • More request of repeating or revoicing for the idea of systematic organization	• Less request of repeating or revoicing after eliciting each proposal • Asking for the collective agreement about each proposed answer • More request of repeating or revoicing for the idea of systematic organization

8.3.2. Refining A Model of Mathematical-Task Specificity of Mathematical Explanation

The previous section provides a glimpse of differences in building mathematical explanation for the four mathematical tasks. In this section, I refine the model of mathematical-task specificity of mathematical explanation by articulating the relationship between the nature of mathematical tasks entailed in mathematical tasks, struggles faced by students, and mathematical supports provided by a teacher. Back to the initial argument that I provided at the beginning of Chapter 8.3, I first provided a basic model of developing mathematical explanation into two parts: mathematical-task generality of constructing mathematical explanation and mathematical-task specificity of constructing mathematical explanation.

As illustrated in Figure 8.1, the box in the middle represents difficulties that students have in constructing mathematical explanation, which are affected by psychological, social, and cognitive issues in constructing mathematical explanation on the one hand but anchored in the nature of mathematical explanation entailed in mathematical tasks on the other hand. To support students' development of mathematical explanation, both the employment of pedagogical supports aligned with psychological, social, and cognitive issues and the deployment of mathematical supports informed by the nature of mathematical explanation are needed. I intentionally use different names between pedagogical supports and mathematical supports because the term "deploy" conveys the meaning of "systematically" or "strategically" whereas the term "employ" conveys the meaning of "making use of" in a more general nuance.

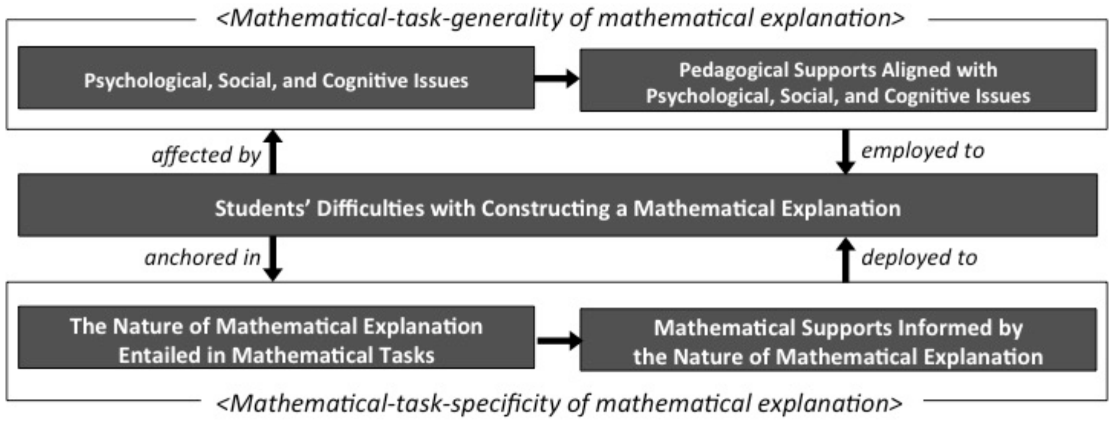


Figure 8.1. A basic model of mathematical-task generality and mathematical-task specificity of constructing mathematical explanation

Figure 8.2 elaborates the mathematical-task specificity of constructing mathematical explanation by specifying the relationship between the nature of mathematical explanation entailed in mathematical tasks, the struggles faced by students in constructing mathematical explanation, and the mathematical supports provided by a teacher. In Figure 8.2, the left column represents the level of mathematical demands entailed in mathematical tasks (being scaled from low to high) and the right column represents the level of mathematical supports provided by a teacher (being scaled from moderate to intense). The four mathematical tasks analyzed in this dissertation study impose different demands in constructing mathematical explanation for the following domains: (1) the level of threshold for accessing the key mathematical ideas; (2) the crucial role of the key incorrect answers; (3) the demand of using accurate language; (4) the demand of using representations; and (5) the demand of organizing solutions spaces. Correspondingly, these different demands embedded in mathematical tasks shape the different levels of mathematical supports provided by a teacher in the following domains: (1) scaffolding during set-up and independent work; (2) preserving the key incorrect answers; (3) surfacing language to articulate; (4) connecting representations to articulate; and (5) having a control of using a public space.

As shown in the Figure 8.2, the two rectangle problems (the brown rectangle problem and the blue and green rectangle problem) are positioned either in the same scale or neighbored together and the two combinatorial problems (the two-coin problem and the three-permutation problem) are positioned either in the same scale or neighbored together. Overall, except two cases, the position of each mathematical task on the scale in the left side column (the nature of mathematical explanation entailed in mathematical tasks) is matched with the positionsof each mathematical task on the scale in the right side column (mathematical supports informed by the nature of mathematical explanation). The reason for positioning the blue-and-green rectangle problem toward the moderate scale rather than the same scale as the brown rectangle problem toward the intense scale for the second row (preserving the key incorrect answers) and the third row (surfacing language to articulate) is the effect of instruction sequence: the students develop the working definition of fraction from the brown rectangle problem, so the need for providing such intense mathematical supports are somewhat alleviated.

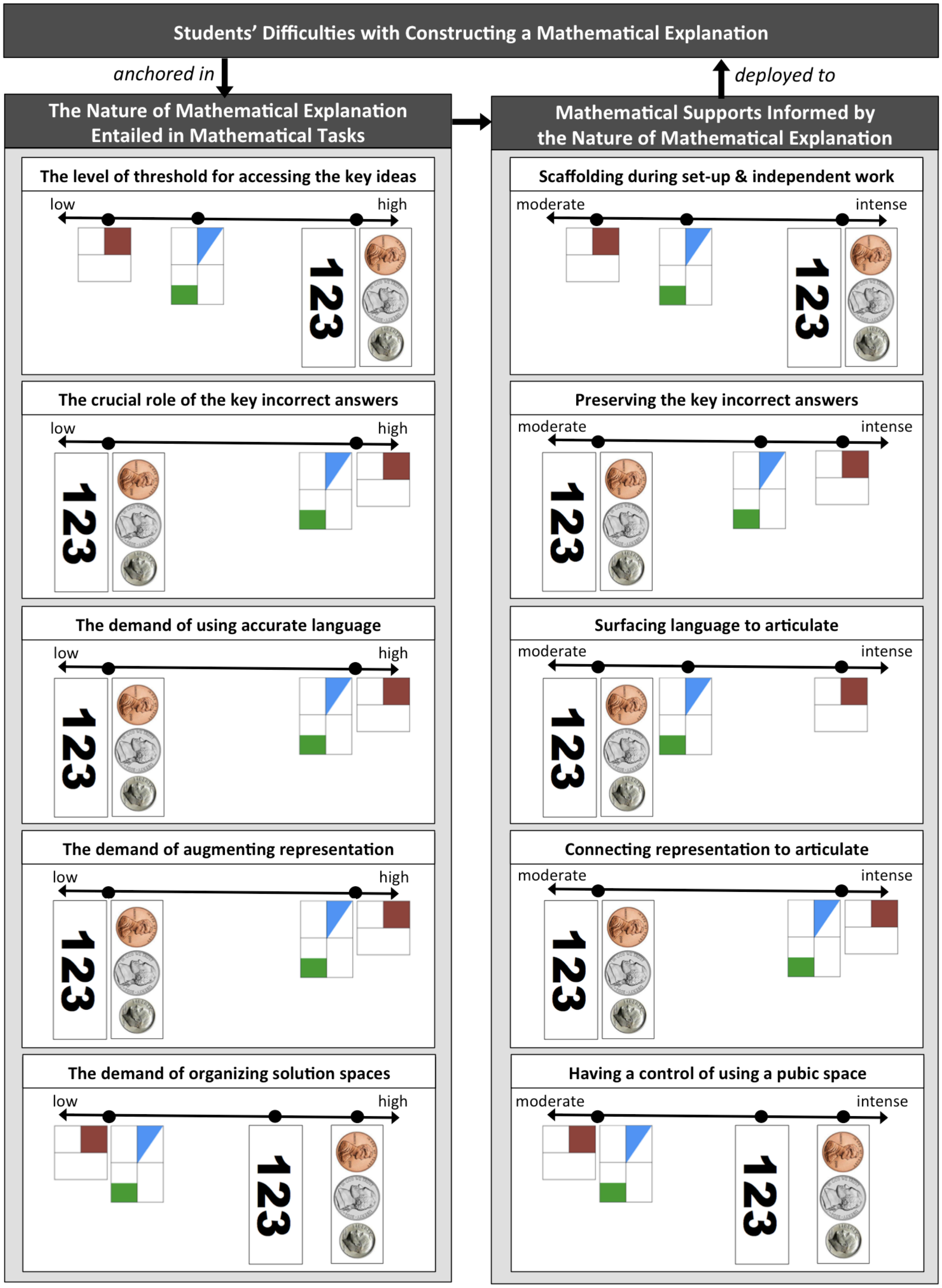


Figure 8.2. An elaborated model of matheamtical-task specificity of constructing mathematical explanation

8.3.3. A Need for Developing the Categorization of Mathematical Tasks Aligned with the Nature of Mathematical Explanation

Mathematical tasks play a crucial role in shaping mathematical activities (e.g., Doyle, 1988; Stein, Grover, & Henningsen, 1996; Stylianides and Ball, 2008) but the categorization of mathematical tasks has not particularly been aligned with the nature of mathematical explanation. The often-used categorizations of mathematical tasks in analyzing curriculum materials or evaluating instructional quality—such as the cognitive demands (memorization, procedure without connection, procedure with connection, and doing mathematics), the number of answers (only one answer; finitely many answers; infinite many answers), the type of responses (answer only vs. explanation required) or the context of the problem (pure mathematical contexts vs. real-life contexts)—contribute to provide rich opportunities for students to engage in worthwhile mathematical tasks, but these categories might not be successfully applied to reveal different demands of mathematical explanation entailed in different mathematical tasks. Among many other categories listed above, this section first examines two cases which pay a closer attention to the practices that approximate mathematical explanation.

The first case is the classification made by Leinhardt (2001). The original context of such proposal is not to make an extensive discussion about different occasions within mathematics but to address "subject-matter-based occasions" that prompt instructional explanations. She illustrates three occasions for instructional explanation in mathematics (actions; principles; and meta-system), making a distinction from four occasions for instructional explanation in history (events; social and political structure; themes across different events and different types of social organizations; and meta-system). The first occasion for instructional explanations in mathematics is actions around operations, functions, procedures, and iterations. In this occasion, she argues that the instructional explanation for operations is constrained by the entities (e.g., whole number vs. fraction) and the representations. The second occasion for instructional explanation in mathematics is principles (e.g., commutative principle, the concept of proof) that form constraints and affordances on entities and operations. The instructional explanation for principles is done by example and logical proof. The last occasion for instructional explanation in mathematics is meta-system such as problem-solving heuristics of simplification, extreme cases, and analogy construction.

As another case, in arguing that knowledge of situations for proving is needed for teachers to mobilize the activity of proving for students, Stylianides and Ball (2008) classify proving-tasks with two mathematical criteria—(1) the number of cases involved in a proving task (a single case, multiple but finitely many cases, and infinitely many cases) and (2) the purpose of a proving task (to verify a statement or to refute a statement). They argue that even in the same mathematical territory the proving tasks can vary by the degree of generality. For example, they illustrate that the task of proving multiples of a number can be varied by the number of cases: "Prove that 186 plus 243 is a multiple of 3" as a single case; "Prove that the sum of any two multiples of 3 between 30 and 50 is a multiple of 3" as multiple but finitely many cases; and "Prove that the sum of any two multiples of 3 is a multiple of 3" as infinitely many cases. They also argue that the different proving opportunities are provided to students depending on different assumption about the sets for the following task: "How many different two-addend number sentences are there for 6? Prove your answer." Depending on different assumptions about the sets, the aforementioned mathematical task has finitely many cases if assuming the set as positive integers and infinitely many cases if assuming the set as integers. Also depending on the statements, they illustrate that students would have opportunities to verify or to refute the statement. They argue that teachers need to "recognize important mathematical differences among these situations, and stage appropriate opportunities to their students to engage in proving" (p.311).

Despite the minor difference in choosing terms (i.e., "occasion" versus "situation"), they both cast the light on the idea of possible differences of mathematical explanation by the mathematical elements. The four mathematical tasks examined in this dissertation study, however, blend more than one element in such a categorization. At a glance, it looks like that all of the four mathematical tasks examined in this dissertation study have finite number of answers: only one answer for the brown rectangle problem and the blue and green rectangle problem but finitely many answers for the two-coin problem and the three-permutation problem. However, as shown in the extensive detailed analysis of instructional interactions in the EML 2010, the brown rectangle problem, in which seemingly has one answer, has actually infinite number of answers if counting equivalent fractions. The key issue here is not to recognize whether the brown rectangle problem has only one answer or infinite number of answers. Rather, it is important to recognize the key ideas of "making equal parts" and "identifying the whole" to develop the definition of fraction. In this sense, the categorization of number of answers does not fit well to characterize the mathematical explanation entailed in the brown rectangle problem.

In addition, all of the four mathematical tasks can be used for both verifying the statement about the answer and refuting the statement about the answer. For example, in case of the blue and green rectangle problem, the mathematical supports needed to verify the statement that "the blue triangle and the green rectangle are the same area" is not quite different from the mathematical supports needed to refute the statement that "the blue triangle and the green rectangle are not the same area." In case of the two-coin problem, the mathematical supports needed to verify the statement of "there are six solutions" would not be extremely different from the mathematical supports needed to refute the statement of "there are three answers" or "there are seven answers." In either case—the verification of the correct number of solutions or the refutation of the incorrect number of solutions—the mathematical work demanded by the two-coin problem is to convince why the proposed solution meets the conditions of the problem and to prove the exhaustiveness of solutions by systematically organizing the solutions.

This dissertation addresses the need for developing the categorization of mathematical tasks that aligns with the nature of mathematical explanation entailed in mathematical tasks. One such approach would be into explanation of definition, explanation of equality, and explanation of exhaustiveness. However, it needs to consider that even within the same occasions for explanation, developing a definition of fraction might be somewhat different from developing a definition of rectangle or developing a definition of even or odd numbers. The explanation of definition, however, might be further differentiated by the need for developing a hierarchical relationship between the concepts and the need for developing a generalizability.

CHAPTER 9.
A CONCEPTUAL FRAMEWORK:
CORE TASKS OF TEACHING
AND INSTRUCTIONAL RESOURCES

9.1. Overview

The purpose of this dissertation is to conceptualize the work of teaching entailed in supporting students to develop mathematical explanation. In Chapters 4 through 7, I provided the detailed analysis of instructional interactions managed by the same teacher for teaching the same mathematical tasks to different cohorts of students sampled from the same school district over years. The initial characterizations elicited from these individual-case analyses become the foundations of building arguments for the cross-case analyses in Chapter 8. The first part of Chapter 8, cross-year analysis, discusses the relationships between instructional features and constructing mathematical explanation. The second part of Chapter 8, cross-mathematical-task analysis, argues for a distinction between “mathematical-task generality” and “mathematical-task specificity” of constructing mathematical explanation. Building on these analytical grounds, this chapter introduces a conceptual framework for the work of teaching entailed in supporting students to develop mathematical explanation around four core tasks of teaching and two instructional resources. This chapter begins with explaining the basic ideas underlying the structure of conceptual framework. It then specifies the details of four core tasks of teaching and two instructional resources to support the work.

9.2. Structure of the Conceptual Framework

The conceptual framework for the work of teaching entailed in supporting students to develop mathematical explanation basically comprises two components: core tasks of teaching and instructional resources. At the early stage of data analysis, beyond a simple distinction between ends (core tasks of teaching) and means (instructional resources), the initial categorization of whether the themes that emerged from the data analysis need to be called as core tasks of teaching or instructional resources were somewhat unclear because of its intimate relationship. In the process of iterative data analysis, in conjunction with an extensive literature review and reflection on my personal experiences of teaching and studying teaching in a variety of settings, however, the analytical distinction between core tasks of teaching and instructional resources has clearly been made. This section begins with sketching out the basic ideas underlying the structure of conceptual framework and then introduces the specific details of the conceptual framework for the work of supporting students to develop mathematical explanation.

First, the main purpose of this dissertation is to conceptualize the work of teaching entailed in supporting students to develop mathematical explanation by decomposing the complexity entailed in the work of teaching into its constituent components. In the study of teaching practices across three different professions (clergy, clinical psychology, and teaching), Grossman et al. (2009) identify decomposition of practice, along with representations of practice and approximations of practice, as the key for understanding the pedagogies of practice in professional education. Grossman (2011) writes:

> For students to learn to engage in complex practice, they must first be able to distinguish the different components that go into that practice. We refer to this work as the "decomposition" of practice—breaking down complex practice into its constituent parts for the purposes of teaching and learning. (…) The ability to decompose practice depends on the existence of a language and structure for describing practice—what we've described as a grammar of practice. Without

> such a grammar, it is difficult to name the parts or to know how the components are related to one another. (Grossman, 2011, pp.2838-2939)

As discussed in Chapter 2, despite the shared interests in decomposition, the grammar of decomposing practice has not been well established, articulated, and discussed yet in research on teaching and teacher education. In decomposing the work of steering instruction toward the mathematical point, Sleep (2009) first identifies seven core tasks of teaching and then illustrates strategies and problematic issues in enacting each core task of teaching, wherein some strategies (e.g., strategically selecting numbers for examples and exercises) are the opposite statements of problematic issues (e.g., nonstrategic selection of numbers in examples or exercises), but does not make a direct connection between each core task of teaching and specific teaching moves. In another example, in decomposing the work of leading a mathematical discussion, Boerst et al. (2011) first identify techniques (e.g., revoicing) and intermediate practices (e.g., clarifying student thinking) and then articulate how a particular practice could be accomplished through different techniques and why a particular technique is used to serve the purpose.

A variety of structures can be employed in decomposing the complexity entailed in the work of teaching: sequential decomposition, chronological decomposition, hierarchical decomposition, multi-level decomposition, and multi-layered decomposition[25], to name a few. One grammatical approach would be to break down the complex work of teaching into pieces by varying the grain size, such as decomposing into the larger grain size of practices for the first-level decomposition and then further decomposing each practice into the smaller grain size of practices for the second-level decomposition. In using such an approach, several questions might be raised: What are the attributes of the decomposed practices? How are the attributes of the decomposed practices in the first-level different from the attributes of the decomposed practices in the

[25] A disciplinary distintion, categorization, and definition about these different decomposition methods has not been made yet. Here, I named different mehtods of decomposition by the relationships between the decomposed practices. One useful apporach might be to think about how the decomposition method addresses the core issue of teaching. More specifically, the chronological deomposition might resolve the issue of temporal order, the hierarchical deomposition might resolve the issue of inclusive relationship, the multi-level decomposition might resolve the issues of nestedness, and the multi-layered decomposition might resolve the issue of simulatneousness.

second level? What should be the granularity of the decomposed practices in the final level? Instead of simply varying grain size across multiple levels, this dissertation takes a different approach by classifying the complexity entailed in the work of teaching into core tasks of teaching and instructional resources. Given that the purpose of decomposition is for teaching and learning practices, identifying useful instructional resources to support learning those practices would be a valuable approach to overcome difficulties entailed in the work and polish skills demanded by the work.

Second, instructional resources are in the service of core tasks but do not directly or automatically cause student learning, particularly for the development of mathematical explanation in this dissertation study. Addressing the need for shifting the research paradigm from the effect of resources access and allocation on student learning to the use of resources in the dynamic instructional interactions, Cohen et al. (2003) argue that resources[26] themselves do not cause learning but moderate the effect of the key causal agents on student learning. Considering that the same access to conventional resources (e.g., class size or curriculum materials) does not lead to the same results for learning, an important question to be asked is how these conventional resources are used by the active agents of instructional interactions. This argument about conventional resources applies to the contexts of two instructional resources examined in this dissertation study: discourse resources and collective resources. In general, discourse moves, such as repeating, revoicing, agreeing, or disagreeing, have been considered as crucial pedagogical strategies and techniques that must be adopted to create a classroom environment that promotes students' active participation and increase their discursive activities. Instead of naming them as strategies or techniques, I argue that these discourse moves are instructional resources because the mere use or the overuse of these discourse moves does not create the productive mathematical learning for students. In a similar vein, a teacher teaches a group of students, not just an individual student, but the composition of the group itself does not lead to the development of mathematical explanation either. Rather, depending on its use, instructional resources of discourse

[26] Cohen et al. (2003) illustrate different types of resources: conventional resources (e.g., class size, curriculum materials, and facilities), personal resources (teacher's and learners' will, skill, and knowledge), and environment/social resources (e.g., state/district guidance for instruction, professional leadership, and family support).

resources and collective resources constrain or enrich the construction of mathematical explanation.

Third, one important distinction made between core tasks of teaching and instructional resources is the extent in which a mathematical task plays a role in shaping its demands. The specific mathematical knowledge and skills required to enact core tasks of teaching are different across mathematical tasks, but the demands of enacting core tasks of teaching are quite similar across mathematical tasks. Unlike core tasks of teaching that are commonly undertaken regardless of mathematical tasks, the role of instructional resources varies by the nature of mathematical tasks. Take a metaphor of teaching and learning different swimming strokes. Regardless of swimming strokes, beginning swimmers need to learn arm movements, leg movements, breathing, and the coordination among them. These could be called core tasks of swimming. Failing to enact any of these core tasks makes it difficult to do the work of swimming effectively and proficiently. On the other hand, different resources could be adopted differently to train specific skills for different swimming strokes. For example, a kickboard is used to strengthen the arm movement by squeezing it between thighs in freestyle stroke and backstroke, but is used to isolate the leg movements by holding it in front of the body in breaststroke. In case of butterfly stoke, it helps beginning swimmers learn the body undulation. Kickboard is useful resource to strengthen skills and to isolate techniques in the multifaceted work of swimming in general, but the specific role played by kickboard and its uses are different across different swimming strokes. As another example, swimming fins are more effective resource for accelerating the flutter kick than kickboard for freestyle stroke and backstroke and more effective for learning body undulation for butterfly stroke, but are not very useful to enhance skills needed for breaststroke.

This metaphor of swimming reflects the idea that the role of instructional resources can also be varied by the nature of mathematical tasks. Similar to the fact that the role of kickboard depends on the type of swimming strokes, the role of specific instructional resource depends on the type of mathematical tasks. The same instructional resources can be used in very different ways, corresponding to the needs for strengthening specific skills that are required to fulfill the nature of mathematical explanation entailed in mathematical task. Depending on the demands of accurate language use entailed in the mathematical task, discourse resources are differently deployed and loaded across mathematical tasks. For example, the discourse moves of repeating and revoicing are heavily loaded at the initial stage of developing mathematical explanation for the brown rectangle problem, but they are less loaded at the initial stage of developing mathematical explanation for the two-coin problem. This observation, along with the mathematical-task specificity of constructing mathematical explanation in Chapter 8, suggests the inclusion of another layer of mathematical task in the conceptual framework.

Lastly, instructional resources are used to serve core tasks of teaching in general, but I do not build a one-to-one correspondence between each instructional resource and each core task of teaching in this dissertation study. Back to the swimming metaphor, it might be useful to specify how each swimming resource could be used to strengthen the enactment of each core task of swimming, but a one-to-one correspondence is not always doable for every resource. For instance, swimming fins dramatically improve the leg movement by accelerating dolphin kick but are not quite associated with improving arm movement or breathing. Instead, I mainly focus on identifying different role of instructional resources shaped by different mathematical tasks.

Figure 9.1 illustrates the conceptual framework for the work of teaching entailed in supporting students to develop mathematical explanation. The components identified in the elaborated model of mathematical-task specificity of mathematical explanation, introduced in Chapter 8, are shaded as gray below. As shown in the top of Figure 9.1, students' difficulties with constructing a mathematical explanation are anchored in the nature of mathematical explanation entailed in the mathematical task. The nature of mathematical explanation shapes the role of instructional resources, which are used to serve the core tasks of teaching, and shapes the level of mathematical supports needed, which in turn are deployed to support students' development of mathematical explanation. As shown in the bottom of Figure 9.1, the level of mathematical supports provided for students to develop mathematical explanation are not uniform across mathematical tasks. I positioned the level of mathematical supports on the specific point of the scale in Chapter 8.3.2 (see Figure 8.2), but represent the level of mathematical supports that are adjustable to the nature of mathematical task as a more generic way in this Chapter (see Figure 9.1).

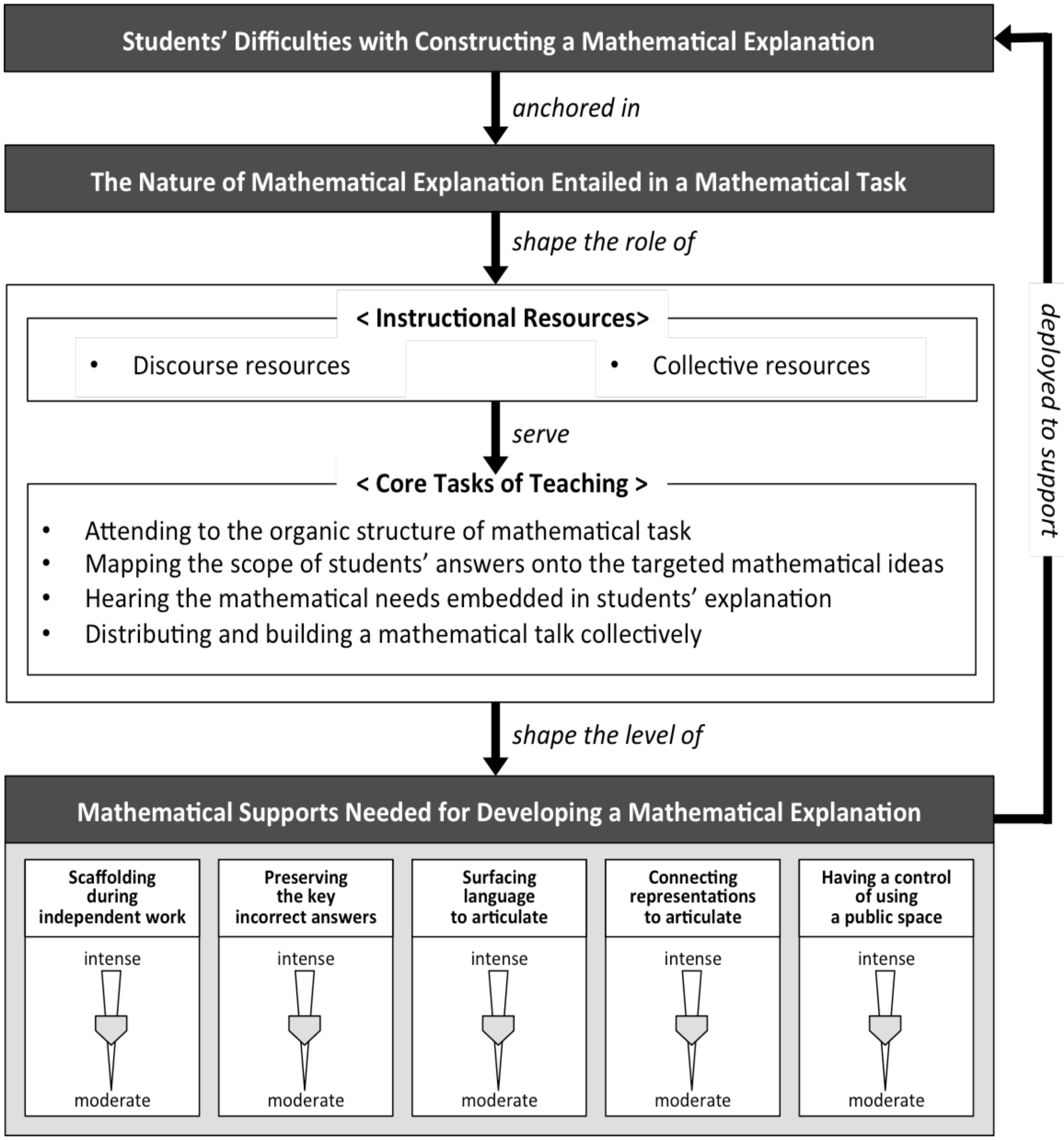


Figure 9.1. A conceptual framework for the work of teaching entailed in supporting students to develop mathematical explanation

9.3. Core Tasks of Teaching

Building on the themes emerged from individual-year analysis, cross-year analysis, and cross-mathematical-task analysis, I develop the following four core tasks of teaching to support students to develop mathematical explanation: (1) attending to the organic structure of mathematical task; (2) mapping the scope of students' answers onto the targeted mathematical ideas; (3) hearing the mathematical needs embedded in students' explanation; and (4) distributing and building mathematical talk collectively. The basic ideas in developing these four core tasks of teaching are as follows.

First, the four core tasks of teaching that I decomposed in this dissertation are not just a random collection of observations conveniently or randomly extracted from the data but deliberately devised to structurally capture the crucial elements of instructional interactions while being consistent with the data. Taking seriously account of the three-pronged arrows that a teacher has relationships in the instructional triangle (students, content, and students-content), each core task of teaching anchors in both mathematics and students. Even though there exist differences in what is in the foreground and what stays in the background, all of the four core tasks of teaching pay attention to the coordination between students and mathematics. As shown in Figure 9.2, these four core tasks of teaching are not temporal sequence, but vary with the degree of complexity regarding to the demand of unpacking students' ideas. However, this does not mean that the core task located on the less complex side is less intellectually demanding whereas the core task located on the more complex side is more intellectually demanding. In fact, attending to the organic structure of the mathematical task is fairly mathematically demanding task which establishes foundations for enacting other core tasks of teaching.

The core tasks of teaching do not portray any particular approach to teaching (e.g., teacher-oriented instruction vs. students-oriented instruction; problem-solving instruction vs. inquiry-based instruction), but deal with the essential aspects of teaching that need to be taken account for student learning. They are not prescriptions but demand substantive mathematical knowledge to carry out each core task of teaching. Approaching through pedagogical strategies or techniques would be one way of examining what is entailed in supporting students to develop mathematical explanation, but it increases a risk of losing one of these crucial elements of instructional interactions.

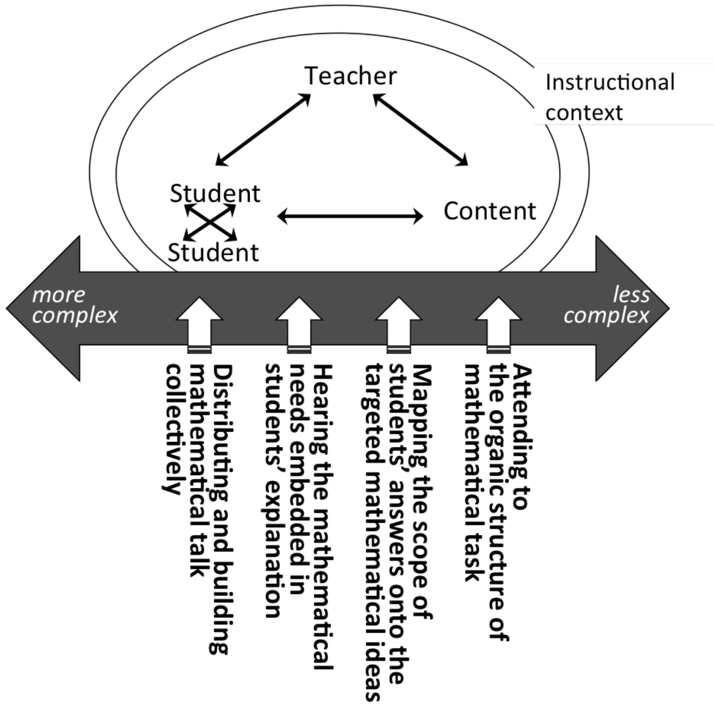


Figure 9.2. Four core tasks of teaching anchored in the instructional triangle while gradually increasing the degree of complexity

Second, each core task of teaching is further articulated into problems that hinder the enactment of the core task of teaching. In doing so, this dissertation avoids terms such as "strategies" or "techniques" because those terms might (1) convey the nuance of prescriptive or exhaustive lists; (2) raise an issue around whether they are routines or improvisations; (3) be too self-evident to address the need of instructional resources; (4) regress to the idea of the actions taken by a particular expert teacher; and (5) leave out capturing different instances. For such reasons, I translate the skillful management that I noticed from the data to possible problems that hinder the enactment of the core task of teaching. However, the problematic issues could be easily translated back into the skillful management of each core task of teaching or vice versa. Both the statements of skillful management and problematic issues to enact each core task of teaching would be somewhat redundant and unnecessary. Figure 9.3 illustrates the summary of core tasks of teaching and problematic issues that hinder the enactment of each core task of teaching.

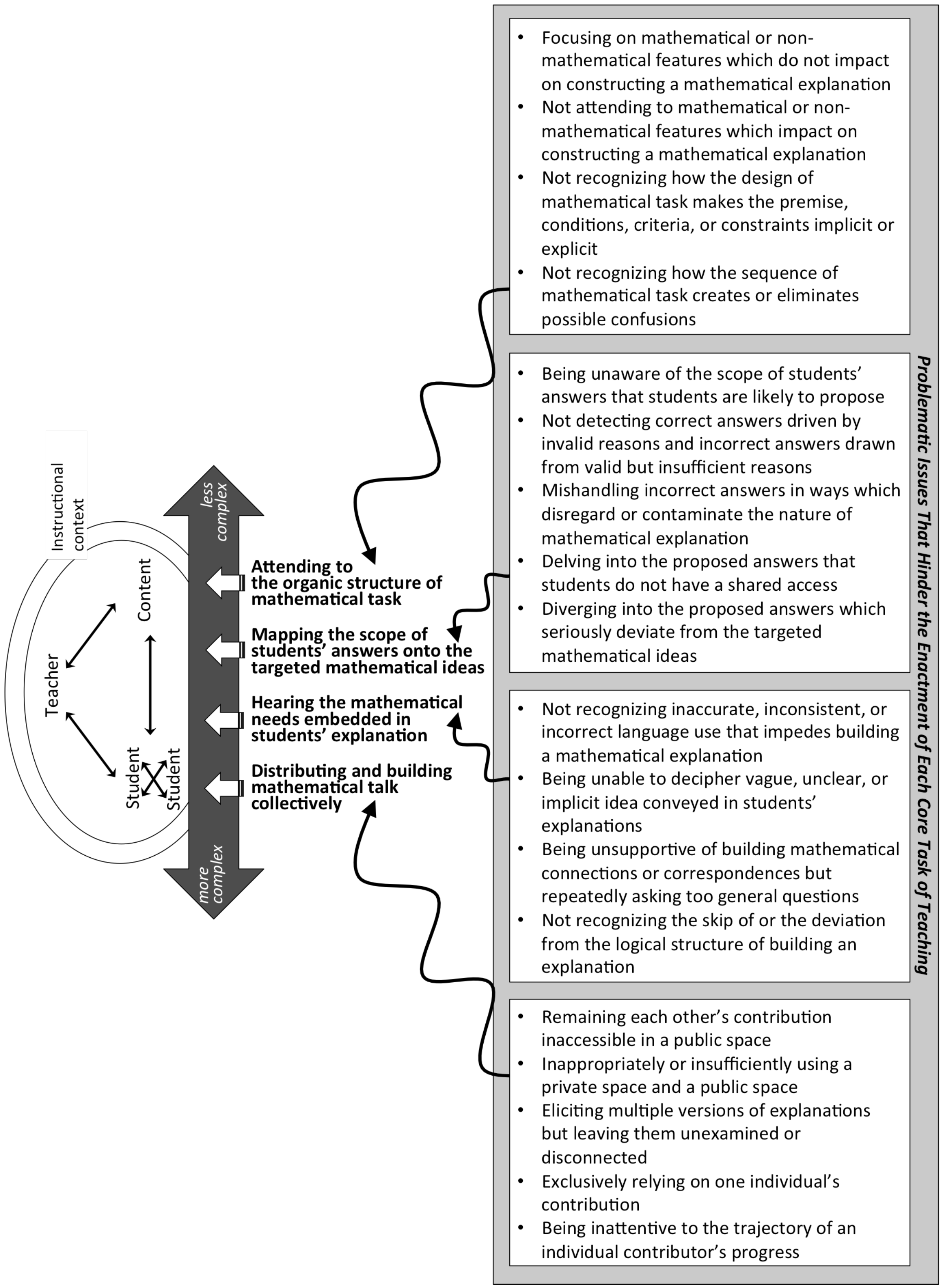


Figure 9.3. A conceptual framework of core tasks of teaching and problematic issues that hinder the enactment of each core task of teaching

Lastly, given that the purpose of decomposition is to make the multifaceted complex work of teaching understandable, learnable, and doable, especially for beginning practitioners, one important consideration would be to examine whether a set of decomposed practices captures the holistic aspect of the work well without losing any crucial element of the work. Recomposing the decomposed practices is a way of validating the scale, accuracy, dimensionality, and quality of decomposition, but it is beyond the scope of this dissertation study. Using the findings from this dissertation, recomposing the decomposed practices could be carried out in teacher education programs or professional development programs. By identifying areas which are not handled or covered by these decomposed practices, the current conceptual framework to support students to develop mathematical explanation could be further improved.

In the following sub-sections, I provide the details of what each core task means and the problematic issues that hinder the enactment of each core task of teaching.

9.3.1. Attending to the Organic Structure of Mathematical Task

As the nature of mathematical explanation is largely shaped by mathematical task, an important core task of teaching is to figure out the organic[27] structure of mathematical task. By the organic structure of mathematical task, I refer to the fundamental, interrelated, and integral elements that cohesively constitute the structure of mathematical task. As indicated by the term "organic," if one of such elements is missing or malfunctioning, this limits the capacity, potential, and terrain of mathematical task. Attending to the organic structure of mathematical task basically concurs with the idea embedded in one of the themes for disciplinary analysis of teaching and learning mathematics in classrooms proposed by Thames (2009): the structure of problems and the role of problems. He writes:

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[27] American Heritage Dictionary defines "organic" as "of, relating to, or derived from living organisms," "of, relating to, or affecting a bodily organ," "having properties associated with living organism," and "constituting an integral part of a whole; fundamental." The term of "organic structure" is more widely used in describing the features of organization in business. In contrast to "mechanic structure," "organic structure" is used to highlight the joint specialization, complex integrating mechanism, and mutual adjustment in business.

> It would seem that the activity of mathematical research is, in part, about being vigilant to what the [mathematical] tasks are, how they shift, and how they are related, even if they do not typically use the language of "[mathematical] tasks" per se. This is a distinctive disciplinary attention to mathematics problems that goes well beyond the usual attention given to school mathematics problems in other kinds of analyses. (Thames, 2009. p.121)

The specific sub-categories of the structure of problems and the role of problems proposed by Thames (2009)[28] are not the same as the problems that hinder the enactment of this core task of teaching identified in this dissertation, but my "distinctive disciplinary attention" resonates with Thames' study (2009). In this sense, the organic structure of mathematical task goes beyond the commonly used features to define mathematical task such as the number of answers, the contextual features (i.e., purely mathematical context vs. real-life context), the level of difficulty (i.e., easy vs. hard), and the cognitive demands (i.e., memorization vs. doing mathematics) as well as general features of problem statement that asks for explanation (e.g., "explain your reasons").

Along the spectrum of the four core tasks of teaching illustrated in Figure 9.3, attending to the organic structure of mathematical task is located in the least complex side in the student-content arrow due to its relatively low demands in unpacking students' ideas. As I addressed in the previous section, however, this does not mean that this core task is the least intellectually demanding. Rather, this core task of teaching demands substantial mathematical knowledge and sets the foundation for enacting other core tasks of teaching. It is likely that this fundamental core task of teaching is often unnoticed by teachers because they often teach mathematical tasks just as written in curriculum materials rather than devoting their efforts to critically analyze how the structure of mathematical task promotes the development of mathematical explanation or creates obstacles for the development of mathematical explanation. Without attending to the organic structure of mathematical task, it is less likely to provide substantive and crucial mathematical supports for students to develop mathematical explanation. Much of the

[28] The five sub-categories are (1) noticing what the tasks are, when they start and stop, and similarities and differences among tasks; (2) identifying sub-tasks and how they are related to the larger task; (3) noticing when the criteria or constraints of a problem are changed or refined; (4) expressing tasks in terms of their basic mathematical structure or mathematically salient features; and (5) evaluating the mathematical difficulty of a task. (Thames, 2009, p.121)

arguments about mathematical approaches discussed in Chapter 8.2.3 are related to this core task of teaching. The following four problems hinder the enactment of attending to the organic structure of mathematical task.

The first problem is focusing on mathematical or non-mathematical features which do not impact on constructing a mathematical explanation. It is easy for teachers to deviate from the key ideas embedded in mathematical task by focusing on less critical, relevant, or crucial features that do not greatly contribute to the construction of mathematical explanation. To overcome such a problem, mathematical diagnosis would be needed to filter subsidiary issues that prevent, delay, or obstruct the construction of mathematical explanation. Take as an example of the two-coin problem discussed in Chapter 6. Promoting multiple representations and sharing different representations are generally important in teaching mathematics, but investing too much instructional time on different ways of representing the combination of two coins does not greatly contribute to develop mathematical explanation of proving the exhaustiveness of answers. Rather, inconsistently recoding the combination of two coins in a public space makes it difficult to see the structure of multiple solutions which is a crucial procedure to explain the exhaustiveness of six solutions. Acknowledging the importance of using multiple representations in general, however, this domain of mathematical work might be conflicted with supporting the development of mathematical explanation. Spending substantial amount of instructional time on translating the mathematical task into real-life stories, using pictures of coins, and discussing historical figures engraved in coins increases students' motivation on the mathematical task but has a risk of delaying the construction of mathematical explanation. The probability of drawing each combination of two coins (e.g., If there are the same number of pennies, nickels, and dimes, which combination of coins is the most likely to be drawn?) is an important mathematical territory that needs to be seriously examined in probability but is less conducive to develop mathematical explanation of exhaustiveness of six solutions either.

The second problem is not attending to mathematical or non-mathematical features which impact on constructing a mathematical explanation. Beyond the commonly used categories to define mathematical tasks, such as the number of answers, the level of difficulty, or cognitive demands, even minor mathematical or non-mathematical features exert a greater influence on constructing mathematical explanation. Much of the discussions about mathematical approaches discussed in Chapter 8.2.3 address this issue. To reiterate, in case of the brown rectangle problem, the availability of sticky line contributes to overcoming the issue that adding a line changes the problem and to resolve the issue of irreversibility to the original problem after drawing a line; to see that drawing a line is a mean to make it easy access to see the equal parts rather than a determinant to name a fraction; and to think flexibly with adding an extra line or removing an existing line that leads to equivalent fractions. In case of the blue and green rectangle problem, the availability of sticky blue triangles and sticky green rectangles contributes to overcome the issue of dissecting the whole into different shapes by drawing lines and to increase the accessibility of direct comparison between the blue triangle and the green rectangle by geometrically transforming one shape into the other. Drawing the rectangles on the grid poster is useful to prove that the parts have equal areas by counting the little units for the brown rectangle problem but complicates the proof because of the shapes of triangle for the blue and green rectangle problem. In the problem statement of brown rectangle problem, the choice between “the big rectangle” and “the rectangle” might create different mathematical opportunities of paying attention to the relative size of different rectangles inside the whole rectangle in which leads to take a different whole rather than the intended whole. In case of the two-coin problem, the use of coins influences the solution spaces and the grounds of their explanation. Not attending to these mathematical or non-mathematical features misses an opportunity to provide crucial mathematical supports for students to develop mathematical explanation.

The third problem is not recognizing how the design of mathematical task makes the premise, conditions, criteria, or constraints implicit or explicit. The degree of explicitness regarding to mathematical premises, conditions, criteria, or constraints is not consistently set up across mathematical tasks. Take an example of the three-permutation problem. Three versions of the three-permutation problem are used: the three-digit number, the three-kids race, and the three-car train. Even though they are mathematically isomorphic (Bass, in preparation), the explicitness of conditions embedded in each mathematical task is not the same. More specifically, the three-digit problem, by nature, excludes the possibility of nominating “1” or “12” as an answer if a clear distinction is made between "three-digit" and "digit three" but the construction of mathematical explanation might be delayed by the nomination of answers which uses the same digit more than once such as "112" or "223." On the other hand, the three-kids race problem entails the condition of using each student exactly once thus reduces the possibility of nominating the same kid repeatedly such as "Maria, Maria, and James" as an answer.

Take another example of the brown rectangle problem. A mathematical hierarchy exists between the first problem (equally partitioned rectangle) and the second problem (unequally partitioned rectangle). Obviously, the first problem has a lower level of difficulty and requires a lower level of cognitive demand to solve the problem because the key idea for naming a fraction (i.e., making equal parts) is already given in the first problem but this idea needs to be discovered by students in the second problem. For this reason, one might simply approach these two sub-problems hierarchically in that the first problem merely serve as an easy entry point to name a fraction and the second problem deepens the conceptual understanding about naming a fraction. On a closer examination of students' written explanations, however, despite the fact that most students get the correct answer for the first problem but less than half of students get the correct answer for the second problem across years (except the EML 2009 cohort, most of them produce the correct answers for both of the problems), they produce more complete mathematical explanation in the second problem than in the first problem. In this sense, there exists a mutually complementary relationship between these two sub-problems in supporting the development of mathematical explanation. The similar geometric appearances between the first problem and the second problem make some students translate the answer and the reasoning from the first problem to that of the second problem. The first problem is often used as a reference in proposing the incorrect answer of the second problem and the second problem in turn makes explicit the often taken-for-granted or unnoticed idea of "equal" for the first problem. Across years, either the teacher or the student has used the explanation for the correct answer of 1/3 for the first problem ("there are three parts and one is shaded") to propose the incorrect answer of 1/3 for the second problem ("there are three parts and one is shaded."). In addition, depending on the constraints on adding the number of lines (one line vs. multiple lines), the brown rectangle problem could be seen as only one correct answer if not considering equivalent fractions or as infinite number of answers if counting equivalent fractions. This observation concurs with Styalinades and Ball (2008) that different assumptions about the proving task create different proving opportunities for students.

The last problem is not recognizing how the sequence of mathematical task creates or eliminates possible confusions in constructing a mathematical explanation. A mathematical task is not an exclusively independent entity, but situated in the sequence and array of other mathematical tasks. The instructional sequence of both relevant and irrelevant mathematical tasks creates some confusions and obstacles with building a mathematical explanation. Depending on the sequence of mathematical tasks, students either correctly or incorrectly transfer the established mathematical knowledge from the previous mathematical tasks. As discussed in Chapter 8.2.3, in case of the two-coin problem, the discussion about writing number sentence for 10 creates confusion about two terms (e.g., 4 pennies plus 4 nickels) with two coins (e.g., 2 pennies). Similarly, in case of the three-permutation problem, the Train problem part I creates confusions about what makes a different train—a different solution means a different size of train regardless of its arrangement for the Train problem part I, whereas a different solution means a different order of arrangement with the same size of train for the three-permutation problem. Being unable to recognize the possible influence of the sequence of mathematical task makes it difficult to provide sufficient mathematical supports for students to develop mathematical explanation.

In summary, the enactment of attending to the organic structure of mathematical task is hindered by (1) focusing on mathematical or non-mathematical features which do not impact on constructing a mathematical explanation; (2) not attending to mathematical or non-mathematical features which impact on constructing a mathematical explanation; (3) not recognizing how the design of mathematical task makes the premise, conditions, criteria, or constraints implicit or explicit; and (4) not recognizing how the sequence of mathematical task creates or eliminates possible confusions.

9.3.2. Mapping the Scope of Students' Answers onto the Targeted Mathematical Ideas

Another core task of teaching is mapping the scope of students' answers onto the targeted mathematical ideas. By the scope of students' answers, I refer to the likely answers that students come up with in solving a mathematical task. As indicated by the term "scope," it is not just restricted to the collection of correct answers solely defined by the given mathematical task but open spaces to take account of students' thinking. Mapping the scope of students' answers onto the targeted mathematical ideas basically concurs with the idea embedded in one of the themes for disciplinary analysis of teaching and learning mathematics in classrooms proposed by Thames (2009): the structure of answers and the role of answers. He writes:

> Answers are common currency in mathematics classrooms. However, attentive listening for answers from a disciplinary perspective suggests more, mathematically speaking, than judging whether answers are right or wrong. It is about listening for potential answers, for characterizing those answers in more general terms, and for moves that hold promise for advancing a viable solution. (…) Similar to the work on identifying [mathematical] tasks, a disciplinary analysis of answers is concerned with the structure of answers, salient features of answers, and the structure into which those answers fit. (Thames, 2009, p.125)

The specific sub-categories of the structure of answers and the role of answers proposed by Thames (2009)[29] are not the same as the problems that hinder the enactment of this core task of teaching identified in this dissertation, but my "distinctive disciplinary attention" resonates with Thames' study (2009). Over the decades, researchers have made efforts to utilize problems with multiple correct answers rather than problems with only one correct answer on the one end and they have paid attention to incorrect answers as a main target of public discussion rather than as an object to be eliminated, avoided, or eschewed on the other end. Beyond the judgment about the potential enrichment of mathematical explanation based on the multiplicity of correct answers or the presence of

[29] The five sub-categories are (1) noticing answers; (2) expressing answers in terms of their basic mathematical structure or mathematically salient features; (3) noticing when there are multiple solutions and the mathematical structure of the solution space; (4) noticing when solutions to a problem have a similar mathematical structure with solutions to some other problem that occur earlier, later, or that might have occurred; (5) noticing mathematical moves that advance the correct answer. (Thames, 2009, p.124)

incorrect answers in a public space, this core task of teaching considers the mathematical status of these correct answers and incorrect answers. All of the correct answers do not have the equal status to be fully discussed in a whole group setting. Depending on the accessibility to the shared knowledge, the correct answers are treated differently. Likewise, all of the incorrect answers do not have equal status to be fully discussed in a whole group setting. Depending on the alignment with the targeted mathematical ideas, the incorrect answers are also treated differently. Due to these two features—the accessibility to the shared knowledge and the alignment with the targeted mathematical ideas—, this core task of teaching goes beyond the practice of anticipating and the practice of selecting proposed by Stein et al. (2009).

Along the spectrum of the four core tasks of teaching illustrated in Figure 9.3, mapping the scope of students' answers onto the targeted mathematical ideas is located in the second least complex side in the student-content arrow due to its relatively lower demands in unpacking students' ideas. Compared to the previous core task of attending to the organic structure of the mathematical task, however, this core task of teaching increases the demands of taking account of students because it involves anticipating what students are likely to come up with in solving a mathematical task and interpreting mathematical grounds underneath their proposed answers. The following five problems hinder the enactment of mapping the scope of students' answers onto the targeted mathematical ideas.

The first problem is being unaware of the scope of answers that students are likely to propose. The importance of anticipating answers has been well addressed by Stein et al. (2009) in orchestrating a productive whole-group mathematical discussion. Being unaware of the scope of answers that students are likely to come up with and being mathematically unprepared to deal with those proposals in advance make the management of instructional interactions more challenging. Teachers would have some sense about the scope of students' answers based on their experiences of teaching over years or through conferring with other teachers, but they could also figure out the scope of students' answers by assigning the mathematical task as homework in advance or by surveying the scope of answers while circulating the classroom. In addition, conducting a mathematical analysis of the mathematical task and experimenting with what kind of key ideas students are likely to misunderstand are beneficial for anticipating the scope of students' answers. For instance, for the brown rectangle problem, the two key ideas are "identifying the whole" and "making equal parts" in the area model of part-whole relationship. This mathematical analysis leads one to anticipate that students are likely to misidentify the whole (e.g., the incorrect answer of 1/2 for the second part of the brown rectangle problem) or they are likely to miss the equal partition (e.g., the incorrect answer of 1/3 for the second part of the brown rectangle problem) or they are likely to attend to part-part relationship in conjunction with missing the equal partition (e.g., the incorrect answer of 1/2 for the second part of the brown rectangle problem). As another example, for the three-digit problem (1, 2, and 3), one of the key ideas is to use the conditions of the problem: (1) three-digit, (2) using each digit only once; and (3) using 1, 2, and 3 only. This mathematical analysis leads one to anticipate that students are likely to use numbers which are not three-digit (e.g., the incorrect answer of 3 or 12) or they are likely to use each digit more than once (e.g., the incorrect answer of 112). Given that the composition of answers that a group of students bring to a whole class—whether correct answers are more prevalent or incorrect answers are more prevalent—is likely to influence the dynamic of instructional interactions and the probes to be employed by a teacher, being aware of the composition of students' answers would be also important to support students to develop mathematical explanation.

The second problem is not detecting correct answers driven by invalid reasons and incorrect answers drawn from partially valid but insufficient reasons. Often, invalid reasons are not examined masked by correct answers and valid reasons with some underdeveloped ideas lose the chance to be refined further. This problem resonates with the findings by Erlwanger (1973) in studying Benny's invented but invalid method masked by his correct answers in solving problems. Take an example of the brown rectangle problem. It is easy to miss the chance to investigate whether some students propose the incorrect answer of 1/2 for the second part of the brown rectangle because they misidentify the whole or because they consider the part-part relationship in a conjunction with missing the equal partitioning instead of part-whole relationship. As another example, in the blue-and-green rectangle problem, the further request to explain the incorrect answer of 1/5 reveals that Ethan correctly identified 1/2 of 1/4 but just does not have access to deal with "a fraction of a fraction" algebraically. The detection of an incorrect answer with partially valid but insufficient reasoning creates an opportunity to refine the explanation further. The correct answers driven by invalid reasons or the incorrect answers drawn from partially valid but insufficient reasons can be detected by asking students to write an explanation in their notebooks.

The third problem is mishandling incorrect answers in ways which disregard or contaminate the nature of mathematical explanation. It is worthy noticing that how incorrect answers are used differently to support students to develop mathematical explanation across mathematical tasks. In case of the brown rectangle problem, the incorrect answers are preserved until making a comparison to the correct answers, which leads to the development of a key mathematical idea embedded in the mathematical task. On the other hand, in case of the blue and green rectangle problem, the incorrect answers are followed by making a reference to the definition of fractions first and then compared to the correct answers. In case of the two-coin problem and the three-permutation problem, the incorrect answers are filtered through the examination of whether they fit to the conditions of the problem. The early elimination of incorrect answers for the brown rectangle problem misses the mathematical opportunity to explicitly address "equal partitioning," whereas the preservation of incorrect answers at the later point for the two-coin problem delays the mathematical opportunity to access the threshold of mathematical task. Mishandling incorrect answers that contaminate the nature of the mathematical task hinders the development of mathematical explanation.

The fourth problem is delving into the proposed answers that students do not have a shared access. It is likely that some students propose more "advanced or efficient method" that the classroom community does not have shared access or does not establish such knowledge yet. In case of the brown rectangle problem, several students propose multiplying the same number to the numerator and to the denominator to make equivalent fractions in the EML 2010 rather than explaining equivalent fractions using a pictorial representation. In case of the blue and green rectangle problem, several students approach with the answer as algebraically multiplying fractions rather than geometrically decomposing. For the three-permutation problem, some students explain six solutions using multiplication (i.e., 3×2) but it is not clear how this method (combinatorial rather than multiplicative) is accessible and usable for others. In some sense, these approaches might be more mathematically advanced or efficient methods but are inaccessible for students without substantial additional work and further justification.

The last problem is diverging into the proposed answers which seriously deviate from the targeted mathematical ideas. Not every incorrect answer has the same mathematical status to be discussed. The teacher often asks students whether they agree or disagree with the correct answer, but asks them not to agree or disagree with the mathematically reasonable incorrect answers that are key to developing the targeted mathematical ideas. Not all of the incorrect answers, however, have the same mathematical status to be protected for an in-depth discussion. Take an example of the brown rectangle problem. The teacher makes a very careful call for the incorrect answers which misses the premises of making equal parts for naming a fraction (e.g., 1/3 for the second part of the brown rectangle problem) or which makes a conditional statement about the intended whole for naming a fraction (e.g., 1/2 for the second part of the brown rectangle problem). Unlike her normal calls for not to agree or disagree with these key incorrect answers (e.g., 1/3 or 1/2 for the second part of the brown rectangle problem), the teacher makes a direct call for agreement and disagreement for the incorrect answer which does not need to be protected (e.g., 1/2 for the first part of the brown rectangle problem).

In summary, the enactment of mapping the scope of students' answers onto the targeted mathematical ideas is hindered by (1) being unaware of the scope of answers that students are likely to proposes; (2) not detecting correct answers driven by invalid reasons and incorrect answers drawn from partially valid but insufficient reasons; (3) mishandling incorrect answers in ways which disregard or contaminate the nature of mathematical explanation; (4) delving into the proposed answers that students do not have a shared access; and (5) diverging into the proposed answers which seriously deviate from the targeted mathematical ideas.

9.3.3. Hearing the Mathematical Needs Embedded in Students' Explanation

The next core task of teaching is hearing the mathematical needs embedded in students' explanation. By hearing the mathematical needs, I refer to the attentive listening for disciplinarily translating informal, vague, and incomplete students' ideas into mathematical language, grammar, formulation, and structure. It also includes the discovery of potential ideas that are conducive to expand the mathematical horizon or to deepen the mathematical grounds embedded in students' explanation. Ball (1997) addresses three inherent challenges in listening[30] student talk: (1) listening across divides; (2) listening through the multiple influences of contexts; and (3) listening with and through desire. Among these three inherent challenges of listening, hearing the mathematical needs embedded in students' explanation basically concurs with the idea related to the first challenge of listening: listening through divides[31]. She writes:

> As they work to help their students develop understandings of content, teachers must listen to students—making use of their own knowledge of the content while not being limited by it. A teacher who knows little of the content, or knows it in only narrow and rigid ways, may miss children's often wondrous insights. But, paradoxically, a teacher with considerable depth of knowledge may fail to hear the nonstandard perspective, the novel insight, listening only for 'the answer.' Teachers, argues Hawkins (1974/1972), must be able to notice when children's observations and questions take them "near to mathematically sacred ground' (p.113)—that is, to the edges of wonder or to core foundations. (Ball, 1997, p.775)

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[30] Ball (1997) did not make a strict distinction between hearing and listening, while these two concepts are often distinguished in the contexts of language acquisition and development. That is, hearing involves physical ability without having any intentions, whereas listening involves meaning-making processing through cognitive abilities with having intentions. In the later part of the article, Ball (1997) defines hearing as watching, wherein "reading" involves "integrating multiple media: facial expression and diagrams, words, and body language." In this dissertation, hearing does not restrict to the physical ability to perceive sound by detecting vibrations through ears, but includes more broad concepts of intentionally listening, reading, and watching.

[31] The second challenge is more related to the coordination of collective work. The third challenge is more related to the pedagogical hearing aligned with the instructional goals in a teacher's mind. The alignment with the instructional goal is possibly to infer in part, but as an outsider of the teaching practice, it is implausible to make an argument about how a certain desire shapes the teacher's responses/probes to students' ideas. By matching what a student said and what a teacher said, as an outsider of the teaching practice, I am only able to partially infer how the teacher reconstructs students' talk into more standard form.

This core task of teaching is easily missed by teachers because their insufficient mathematical knowledge makes it difficult to decipher student thinking or their finished form of knowledge makes it difficult to sensitize the unfinished form of student thinking. It is also likely to be missed by teachers due to underestimating the potential value of student talk or failing to trace the unclear mathematical path underneath student explanation. As I addressed earlier, this core task of teaching relates to the disciplinary translation of student explanation in a more disciplinary meaningful way but does not prescribe specific moves, responses, questions, prompts, and probes that should be employed by a teacher. Thus, this core task of teaching does not specify how the specific pedagogical desire formulates the type of listening such as receptive, sympathetic, generous, skeptical, and critical listening.

Along the spectrum of the four core tasks of teaching illustrated in Figure 9.3, hearing the mathematical needs embedded in students' explanation is located in the second most complex side in the student-content arrow due to its relatively higher demands in unpacking students' ideas. This core task of teaching is located in a certain distance from the content vertex of instructional triangle, but still needs substantial mathematical knowledge to decipher, translate, unpack, and scale up student talk. The following four problems hinder the enactment of hearing the mathematical needs embedded in students' explanation.

The first problem is not recognizing inaccurate, inconsistent, and incorrect language use that impedes building a mathematical explanation. The importance of using accurate, consistent, and correct language might be underestimated by teachers because the terms chosen by students are likely to be masked by the correctness of answer. Or it is possible that teachers might not recognize how the use of inaccurate, inconsistent, and incorrect language can be further misinterpreted by another student that interrupts the development of mathematical explanation in a more coherent way. Because of its sensitivity to language, one ambiguous word chosen by an initial student could be inaccurately taken up by another student, which turns into pandemonium in managing the dynamic interactions and proliferates the complexities of mathematical issues to be resolved. For instance, in case of the brown rectangle problem, students use inaccurate and incorrect terms such as "even" and "half" to indicate the need for drawing a line to make equal partitioning. The term “even” is often used in daily life to indicate the equal sharing but it might conflict with the mathematical definition of “even” numbers that are divisible by two without a remainder. The term “half” is used to indicate the equal partitioning for the right side of the big rectangle and the term “the third box” is used to refer to the right side of the big rectangle, but it might conflict with naming the shaded part as 1/4. Other geometric terms such as “regular rectangle” or “cube” also interferes the construction of mathematical explanation for the brown rectangle problem.

The second problem is being unable to decipher vague, unclear, or implicit ideas conveyed in students’ explanations. It is often pedagogically and mathematically demanding to decipher what students say. Regardless of the vagueness delivered by students, it is worthy noticing that the publicly aired ideas need to be rigorously inspected by students and a teacher. In analyzing the data, I noticed that, surprisingly, students have great capacities and potentials to decipher their colleagues’ somewhat incomplete form of knowledge and vague talk. For instance, in case of the brown rectangle problem, it is not easy to detect what Jaclyn refers to “what if there’s no pictures?” at the first place but it is later elaborated to “what if there is no line in the picture?” which triggers the idea of deleting the existing line or the idea of adding additional line that results in unequal partitioning in the EML 2010, which advances the idea of making equivalent fractions.

The third problem is being unsupportive of building mathematical connections or correspondences but repeatedly asking too general questions. Teachers often adopt general questioning strategies such as “Could you explain?” or “Why?” to support students to elaborate their explanations, but these general questioning strategies are not always sufficient to develop mathematical explanation. In case of the brown rectangle problem and the blue-and-green rectangle problem, the supports to build a connection between the answer and the pictorial representation and the supports to build a connection between the verbal explanation and the pictorial representation are provided by inviting students to the board and by asking them to point out the pieces what they are referring to accompanied by the verbal explanation. In case of the three-permutation problem, instead of just asking general questions, the probes are customized to the intrinsic mathematical structure of the problem to find the structure of the exhaustiveness of six solutions. More specifically, after initially eliciting six solutions, the probe of "what would you do next?" was employed for the three-digit number problem, whereas the probe of "which race is this one most like?" was employed for the three-kids race problem.

The last problem is not recognizing the skip or the deviation from the logical structure of building a mathematical explanation. Because of the features of verbal explanation, it is often easy for students to skip or deviate from the logical structure of the explanation and to jump into other related or unrelated issues. In case of the brown rectangle problem, it is often observed that the students explain the product of reasoning (i.e., "four squares and one shaded") instead of specifying what they have done to get such product (i.e., "drawing a line") and why they need to do that (i.e., "to make equal parts"). Recognizing what to skip and what to deviate from the logical structure is conducive to the development of mathematical explanation.

In summary, the enactment of hearing the mathematical needs embedded in students' explanation is hindered by (1) not recognizing inaccurate, inconsistent, and incorrect language use that impedes building a mathematical explanation; (2) being unable to decipher vague, unclear, or implicit idea conveyed in students' explanations; (3) being unsupportive of building mathematical connections or correspondences but repeatedly asking too general questions; and (4) not recognizing the skip of or the deviation from the logical structure of building a mathematical explanation.

9.3.4. Distributing and Building Mathematical Talk Collectively

The last core task of teaching is distributing and building mathematical talk collectively. By distributing and building mathematical talk collectively, I mean the development of mathematical explanation through a co-construction of collective work rather than through an offer solely made by an individual student. A complete mathematical explanation might be offered by an individual student at the very beginning of the lesson, but the early offer of this individual intellectual property is not the main focus of this dissertation. Among the three inherent challenges of listening (Ball, 1997), this core task of teaching concurs with the idea related to the second challenge of listening: listening through the multiple influences of contexts. Apparently, supporting the development of mathematical explanation only for an individual is not same as supporting the co-construction of mathematical explanation for the collective work. Ball (1997) writes:

> When teachers seek to construct the curriculum in response to students' ideas and understandings, then moment-to-moment appraisals of student thinking are critical. This is hard work, bounded by the uncertainties of ever knowing what anyone means, and multiplied by the complexity of classroom teaching in which teachers are responsible for groups of students, not just individual learners (Jackson, 1968; Lortie, 1975) (Ball, 1997, p.779-780)

Along the spectrum of the four core tasks of teaching illustrated in Figure 9.3, distributing and building mathematical talk collectively is located in the most complex side in the student-content arrow due to its high demands in unpacking students' ideas while coordinating an interaction with a group of students. However, this does not mean that the mathematical demand is weak for this core task of teaching. It still requires substantial mathematical knowledge to do the work. The following five problems hinder the enactment of distributing and building a mathematical talk collectively.

The first problem is remaining each other's contribution inaccessible in a public space. Through using discourse moves or making connections to representations, the deliberate efforts need to be made to make all of the students' contributions accessible to the audience.

The second problem is inappropriately or insufficiently using a private space and a public space. It is often observed that some teachers are not effectively using a private space and a public space. Opening a whole-group discussion without having an opportunity to work on the problem in a private space limits the intellectual engagement and the participation to few students. On the other hand, having no discussion in a public space but only making students to engage in a private space limits the construction of collective work. Too many interruptions of a whole-group discussion by adding the work in a private space interfere with the construction of mathematical explanation. By using a private space effectively at the beginning of whole-group, reflection on the opposing claims, and repositioning one's position at the end of whole-group lesson, it can create rich opportunities for students to engage in developing mathematical explanation.

The third problem is eliciting multiple versions of explanation but leaving them unexamined or disconnected. As I discussed in Chapter 4, it is noticeable that the teacher mainly elicits one explanation for each proposed answer but makes use of the initial explanation thoroughly by repeating, revoicing, asking questions, and making further comments rather than eliciting various explanations for the same answer. One might think that it is ideal for a teacher to elicit a number of different versions of explanations for the same answer by distributing turns to as many students as possible. In doing so, various explanations for the same answer are held in reserve for the development of mathematical explanation, which has the benefits of offering selectivity, richness, thoroughness, efficiency, and effectiveness, but also might increase the complexity a teacher need to deal with when inspecting each explanation rigorously and managing the relationship among various explanations at the same time.

The fourth problem is exclusively relying on one individual's contribution. The work of supporting students to develop mathematical explanation does not rely on when a complete mathematical explanation is offered by one individual student but when the key mathematical ideas are co-constructed by a collective work. In case of the two-coin problem and the three-permutation problem, all six solutions are not elicited from one individual student. The teacher does not just show one individual student's complete record on the board to show the exhaustiveness of answers at the beginning. Instead, the teacher distributes the elicitation of multiple answers from multiple students.

The last problem is being inattentive to the trajectory of an individual contributor's progress. Giving much attention to the collective work is likely to miss the individual student's trajectory, especially for who produce incomplete explanation or for those who produce incorrect answers. It is quite demanding work to track an individual student's contribution while managing instructional interactions with other 25-30 students in a whole-group setting, but paying insufficient attention to the trajectory of an individual contribution limits the development of mathematical explanation. Given that the initial explanation proposed by students, either for correct answer or for incorrect answer, is quite incomplete, tracking the original contributor is important. In case of the brown rectangle problem, Melody provided an incomplete explanation but correct answer for the first part of the brown rectangle problem ("because there's three parts and only one is shaded in") but elaborates on her own initial explanation ("because they're divided into three squares that are the same amount of space") by incorporating the collective work at the later point in the EML 2009. Especially when the students propose incorrect answers, the teacher does not demand the correction of errors on the spot but after giving sufficient opportunities to elaborate the rationale and hearing others' comments, the teacher returns to the students who proposed incorrect answers and hear how their ideas are changed.

In summary, the enactment of distributing and building mathematical talk collectively is hindered by: (1) remaining each other's contribution inaccessible in a public space; (2) inappropriately or insufficiently using a private space and a public space; (3) eliciting multiple versions of explanations but leaving them unexamined or disconnected; (4) exclusively relying on one individual's contribution; and (5) being inattentive to the trajectory of an individual contributor's progress.

9.4. Using Instructional Resources

This section examines the use of two instructional resources that serve the core tasks of teaching to support students to develop mathematical explanation: discourse resources and collective resources. Instead of naming them as pedagogical strategies or techniques, this dissertation study views them as instructional resources because they are utilized through the social dynamic of instructional interactions but do not always function its role in the context of individual work. In other words, the discourse move of "agreeing or disagreeing" does not function as resources for a one-on-one interaction between an individual student and a teacher during an independent work. Similarly, a student's mathematical idea is not dramatically evolved just for the individual setting. The following sections discuss how these two instructional resources are used to support students to develop mathematical explanation and how their roles are shaped by the nature of mathematical tasks in details.

9.4.1. Using Discourse Resources

To support students' development of mathematical explanation, one commonly used pedagogical strategy is to ask "why" question or to request for explaining reasons. Beyond these general questioning strategies, discourse moves such as repeating, revoicing, agreeing or disagreeing, and commenting have been highly recommended for teachers and widely adopted by teachers to enrich a discussion in mathematics classrooms. Acknowledging the needs for such discourse moves, however, while observing instructional interactions managed by a wide range of teachers in various settings I noticed that a number of teachers often overuse or misuse "agreement or disagreement" for any students' responses but do not actually create opportunities for students to discuss mathematical ideas in depth. Instead of indiscriminately using these discourse moves regardless of its mathematical context, the deliberate use of discourse moves aligned with the mathematical issues is needed to support students to develop mathematical explanation.

The discourse move of repeating or revoicing plays a role of: (1) strategically using time when students come up to the board for recording or when a teacher makes a record in a pubic space; (2) making students' explanations available so that the explanations could be further utilized, expanded, and critiqued; (3) making student's language shareable so that they could establish a common ground of mathematical ideas that needs to be parsed out, elaborated, or revised; (4) keeping the mathematical point of given explanation before diverging into other mathematical issues; and (5) checking mathematical understanding, especially when no other tools (e.g., manipulatives, illustrations) are available to use; and (6) if no comments are made, making a quick transition to diagnose whether students' explanation are not available to use or whether discussable ideas are not on the surface yet.

The discourse move of agreeing or disagreeing plays a role of: (1) having collective confirmation about the established mathematical ideas before advancing to the next step; (2) legitimizing disagreement in order to invite incorrect answers into a discussion; and (3) protecting the key mathematical ideas on the table; (4) provoking the debate about opposing claims; and (5) differentiating the context of discussion depending on mathematical issues on the table. The discourse move of commenting plays a role of: (1) addressing issues when the given explanation is not available or understandable; (2) positioning perspectives with the proposed mathematical idea; and (3) creating more open spaces for sharing perspectives, extending ideas, connecting ideas, suggesting new ideas, and evolving ideas into other mathematical territory.

In comparing the use of discourse moves across mathematical tasks, the discourse moves of repeating or revoicing are heavily loaded at the initial stage of developing a mathematical explanation for the brown rectangle problem whereas they are less loaded at the initial stage of developing a mathematical explanation for the three-permutation problem. In case of the brown rectangle problem, the key mathematical idea ("equal parts") is often unrecognized or not explicitly verbalized in spite of recognizing that drawing a line makes equal parts in the unequally partitioned rectangle (the second part of the brown rectangle problem). The discourse move of repeating or revoicing contribute to drawing attention to language and contribute to sharing language. In addition, by repeating or revoicing, it is possible to check whether students hear other students' explanations. On the other hand, the key mathematical idea could be developed by paying attention to the organization of solution spaces for the three-permutation problem, so the discourse moves of repeating or revoicing are less loaded at the initial stage of developing mathematical explanation for the three-permutation problems. In case of the three-permutation problems, it is possible to check students' understanding through available mathematical resources, such as asking for building Cuisenaire rods based on what they heard, asking for proposing another solution according to the structure that other students created, or checking the conditions of the problem. The discourse moves of repeating and revoicing play a more critical role for checking mathematical understanding because no other tools (manipulatives, application of mathematical structure) are available to highlight the key concept in the brown problem, whereas they play a less critical role for checking mathematical understanding because students have tools to build what they heard or they propose another solution based on what they heard in the three-permutation problem.

9.4.2. Using Collective Resources

This section first briefly examines limitations of developing a mathematical explanation by an individual and then benefits of developing a mathematical explanation collectively by a group of students. It follows by addressing possible challenges that a teacher might face in supporting students to develop mathematical explanation collectively in a whole group setting. After that, I explain how the collective work contributes to enrich the development of mathematical explanation, propose a model of how the collective mathematical work is processed toward developing a mathematical explanation, and discuss how the role of collective resources is shaped by the nature of mathematical tasks.

From a cognitive psychological perspective, the effect of self-explanation had been examined in various subject matters in 1980s and 1990s (e.g., Chi et al., 1989; Chi et al., 1994). Regardless of whether self-explanation is spontaneously produced or it is promoted to generate, studies show that verbalizing an explanation is more cognitively beneficial for learning than keeping it just as internal language. Despite this benefit, a number of limitations exist in developing a mathematical explanation just by an individual even with the supports provided by a teacher.

To illustrate this, I return to the episode between Qayshawn and the teacher during individual work at the beginning of the lesson for the brown rectangle problem in the EML 2010, which were analyzed in-depth in Chapter 4. While circulating the classroom during an independent work, the teacher noticed that Qayshawn only produced a written explanation for the first part of the brown rectangle problem but did not have a written explanation for the second part of the brown rectangle problem even though he produced correct answers both for the first part of the brown rectangle problem and the second part of the brown rectangle problem. The initial prompts that the teacher provided was to encourage Qayshawn to write an explanation for the second part of the brown rectangle problem, but Qayshawn changed his correct answer (1/4) to incorrect one (1/3) followed by the teacher's continuous probes. Under the absence of collective resources that Qayshawn can use by hearing other students' repeating and receiving comments from his peers but just being questioned by the teacher in a one-on-one interaction setting, the sustained challenges posed by the teacher seem to give an impression to Qayshawn that he might be wrong. In an individual setting, the lack of opportunities to appropriate, model, appraise, and refine language by hearing other students, the insufficient time to reflect on his or her own thinking while exchanging turns with a teacher, and the absence of stimuli to evoke mathematical thinking limits the potential of enriching the development of a mathematical explanation, especially for developing a definition of mathematical concepts.

One of instructional resources that either a teacher or students can use to develop mathematical explanation is the idea that a group of students collectively constructs, elaborates, and expands. The collective process of generating, reviewing, and certifying ideas is key for constructing knowledge in the discipline. Likewise, a group of students, not just an individual student, collectively creates opportunities to enrich the development of mathematical explanation in classrooms. The collective resources are more than the simple sum of individual's intellectual property, but are synergically constructed through the process of making sense of, interpreting, commenting on, challenging each other's idea, and defending their own ideas. In this sense, the collective work is beneficial in enriching the development of a mathematical explanation.

Despite these benefits, however, teaching a group of students is often considered as challenges rather than resources in teaching. This challenge is well addressed by Lampert (2001) in discussing problems in teaching while leading a whole-class discussion. She writes:

> As I interact with the whole class at once, I need to maintain overall coherence while drawing different kinds of individuals into a common experience of the content. I do this, in part, by calling on students to say something that will contribute to the common experience of the class and then constructing responses to what they say. Equally important are the actions I need to take to engage those students who are not verbal participants, such as drawing on the board. (Lampert, 2001, p.143)

As she addresses, it is difficult for teachers to manage issues around whom to call and what moves need to be followed by; how to coordinate the engagement between one student and the rest of the class; and how to keep the targeted mathematics ideas on track but allow students to make contributions. Given that students reside in different mathematical territory with diverse mathematical disposition and the mathematical terrain changes as students interact with one another in a whole-group setting, creating a common and coherently organized mathematical experience is more challenging for teachers. Despite these challenges, the effective use of collective resources enriches the development of mathematical explanation.

Now, I illustrate how the collective work contributes to enrich the development of mathematical explanation. The analytical approaches I adopted here are (1) comparing the process of constructing a mathematical explanation by an individual with the process of constructing a mathematical explanation through collective work; (2) keeping track of how an individual student's ideas are evolved over time with a dynamic interaction with other students on an individual level; (3) keeping track of what kinds of mathematical ideas are emerged through a dynamic interaction by a group of students in a collective level; and (4) identifying how to make shareable, usable, and accessible an individual student's idea in a public space. I do not separately discuss them but synthesize in terms of how the collective work enriches the development of mathematical explanation based on the data.

First, the collective mathematical work enriches the development of mathematical explanation by elaborating the details of reasoning. Often verbal explanations offered in a public space have more details than written explanations produced in a private space, even though the verbal explanation has some kind of repetitions, stammering, and hesitations at the beginning. By being sensitive to the audience, an explainer makes the implicit or taken-for-granted idea explicit and supplements the vague reference by building a correspondence with a pictorial representation. Also by attentively listening to an explainer, audience asks for clarification about the implicit or vague ideas.

Second, the collective mathematical work enriches the development of mathematical explanation by establishing common language and by naming mathematical objects or ideas together. Through the process of repeating and revoicing the initial explanation, each cohort of the EML students establishes common language to describe the part of the brown rectangle problem (e.g., parts vs. rectangles vs. boxes). The commonly shared language contributes to resolve issues more effectively and efficiently.

Third, the collective mathematical work enriches the development of mathematical explanation by sharing a way of mathematical thinking that triggers the expansion of mathematical ideas. In the EML 2010, Jaclyn's "what-if" question which makes the equal parts unequal by removing the existing line stimulates Coretta to ask "what-if" question which makes the equal parts unequal by adding the additional line.

Fourth, the collective mathematical work enriches the development of mathematical explanation by strengthening claims and warrants to defend their ideas and by devising examples or counter-examples (altering the original mathematical task but not decreasing the cognitive demands of the task). With the challenges posed by opposing claims, students strengthen claims and warrants to support the claims. As student push on each other's reasons, they create or devise examples to support their positions or counterexamples to challenge others' positions. Here, I used the words "create" or "devise" instead of the words of "give" or "show" because these examples or counterexamples are not evident at the beginning of the individual work but are emerged through the dynamic interaction with other students and evolved over time. They often clarify language, crystalize key mathematical issues that lead to make claims clear and develop into more robust explanations. For instance, in the EML 2008, the strong position that adding a line changes the problem by Manoel stimulate Alexico to devise examples to make a counterargument that adding a line does not change the problem.

A model of collective mathematical work that contributes to enrich the development of mathematical explanation is illustrated in Figure 9.4. Through a dynamic interaction with a group of students, individually owned intellectual property could be cultivated to collectively constructed instructional resources while engaging in the following process: (1) explaining and listening; (2) reviewing through commenting and critiquing; (3) justifying and defending; (4) convincing; and (5) certified. For each process, I specify practices engaged by students and practices engaged by a teacher.

As discussed in Chapter 8.2.3, the final production of available collective resources varies to a certain degree across different groups of students but the process of constructing and using collective resources is quite similar. As the process of reviewing through commenting, critiquing, justifying, and defending gets intense, the collectively constructed instructional resources get richer.

Individually Owned Intellectual Property → **Collectively Constructed Instructional Resources**

Process	***explaining and listening*** →	***reviewing through commenting and critiquing*** →	***justifying and defending*** →	***convincing*** →	***certified***
Practices engaged by students	• Being sensible to audience • Being attentive to an explainer • Appropriating language	• Requesting further information, clarification, or revision • Raising problematic issues • Asking "what if" questions • Translating into other contexts • Building criteria for acceptance	• Understanding opposing claims • Refuting invalidity of opposing claims • Reinforcing the warrants by devising examples or counterexamples • Restructuring the logical flow • Adjusting or readjusting a mathematical structure	• Making the implicit explicit • Building mathematically valid rationale • Reflecting on the arguments • Revising the previously produced argument, if necessary	• Using as axiom
Practices engaged by a teacher	• Checking whether an explanation is heard • Increasing an accessibility to explanation by clarifying vagueness at some degree • Identifying mishearing or mis-communicating • Building correspondence between a verbal representation and a pictorial representation • Not readily picking up the complete idea possessed by a single individual	• Cultivating the habits of constructive doubts • Supporting the maintenance of mathematical stance • Assessing the availability of shared knowledge • Evaluating whether it is mathematically rich comment and it aligns with the targeted mathematical ideas	• Arbitrating the debate by distributing sufficient and equal opportunities to opposing claims • Representing the resolving issues in a public space • Evaluating mathematical sufficiency of the warrants provided	• Keeping track of the original contributor • Creating space for individual reflection • Seeking for a collective agreement and checking for no further objection	• Making accessible for further use

Figure 9.4. A model of collective mathematical work that contributes to enrich the development of mathematical explanation

Lastly, the role of collective resources varies by mathematical tasks. The collective mathematical work enriches the development of mathematical explanation for the brown rectangle problem by making incorrect answers a productive site for discussion in a public space, whereas it enriches the development of mathematical explanation for the two-coin problem by making an easy access to the mathematical entry points. In case of the brown rectangle problem, the collective work serves for appropriating, refining, and elaborating language; for devising examples and counter-examples to support one's claim; and for making incorrect answers as a productive site for making the implicit idea explicit. On the other hand, in case of the two-coin problem and the three-permutation problem, the collective work serves for making accessibility to the entry point of mathematical task and for adjusting the mathematical structure.

CHAPTER 10.
DISCUSSION AND CONCLUSIONS

10.1. Summary of Dissertation

The practice of explanation has been set as one of the main goals for teaching and learning mathematics by professional organizations and state-level initiative (National Council of Teachers of Mathematics, 2000; National Research Council, 2001; Common Core State Standards). Besides its value as the goals that students need to achieve in school mathematics, the practice of explanation plays a significant role as a crucial vehicle for learning, a social construct in discipline, a pivotal moment in teaching, and a researchable moment of teaching.

Despite its widely accepted values and increasingly pressing needs, the practice of explanation has not been well cultivated in mathematics classrooms. One reason might be that it is not easy for teachers to share authorities and responsibilities of explaining; on the other side it is psychologically and socially overwhelming for students to give a public speech, hear others' comments, and defend one's position in a public space. Even if these authoritative, psychological, and social challenges are overcome, the endemic challenges still remain in bridging the gulf between teacher knowledge and student knowledge (Cohen, 2011). Given the deficiency of cultivating the practice of explanation in mathematics classrooms, a large body of literature has suggested for teachers to establish norms (e.g., Yackel & Cobb, 1996), to use discourse moves (e.g., Chapin & O'Connor, 2004), and to adopt pedagogical strategies (e.g., Fravillig, 2001; Stein et al., 2009). These approaches have great contributions to increase students' participations, discursive activities, learning opportunities, and intellectual autonomy in general, but the question remains how to support students to develop the skills of constructing and articulating mathematical explanation that resonate with a more profound, robust, and meaningful way of constructing knowledge in the discipline.

In addition to instructional changes called for teachers but instructional challenges faced by teachers to cultivate the practice of explanation, another important issue to consider is whether indicators used to measure the practice of explanation—such as the amount, length, duration, and frequency of talk—adequately depict the nature of the discipline and closely approximate the richness of mathematical explanation. By conceiving the practice of explanation as a much broader discursive activity, however, these approaches are more likely to miss the process of constructing knowledge in the discipline but are less likely to exclude other noises such as off-task talk, off-topic talk, procedural statement (e.g., "invert and multiply" for dividing fractions), rule-of-thumbs (e.g., "you can't subtract a larger number from a smaller number"), mnemonic (e.g., "PEMDAS (Please Excuse My Dear Aunt Sally)" for memorizing the order of operations; "FOIL (First, Outer, Inner, and Last)" for multiplying binomials), or non-mathematical metaphors. Or, they often capture the depth or robustness of mathematical knowledge that teachers have rather than the students' collective construction of disciplinary knowledge through dynamic instructional interactions.

Beyond the practical needs for supporting teachers to cultivate the practice of explanation in a more disciplinarily profound and meaningful way, the problem that motivates this dissertation study is that the practice of explanation has been often equated with discursive activities but less demanded disciplinary rigor, compared to other mathematical practices such as justification, argumentation, or proof. Thus, this dissertation adopts four specific criteria for conceptualizing a more disciplinary grounded mathematical explanation in the context of classroom community: (1) explicating the key ideas, conditions, constraints, or structure embedded in mathematical task; (2) articulating an explanation in a way that reduces the ambiguity, incoherence, inconsistency, inaccuracy, and unnecessary redundancy by utilizing representations, examples, or counterexamples; (3) grounding the legitimate, valid, and logical form of knowledge in the discipline; and (4) grounding the publicly accessible, available, and acceptable knowledge that are shared in the classroom community. The last two criteria of this conceptualization resonate with the characteristics of proof proposed by Stylianides (2007) but my conceptualization can be differentiated with the following two reasons. First, the mathematical explanation conceptualized in this dissertation is more specific to the nature of mathematical task rather than generic features such as the number of answers (e.g., only one answer vs. multiple answers) or the problem statement (e.g., verifying vs. refuting). Second, this conceptualization somewhat alleviates the requirement of refutation. Indubitably, the refutations made by other students enrich and expand the warrants of mathematical explanation but it is not an indispensable aspect of mathematical explanation. One caution is that this dissertation does not aim to evaluate whether each statement provided by an individual student at a particular instructional moment can be counted or acceptable as mathematical explanation or not (c.f., Yackel & Cobb, 1996). Rather, this dissertation focuses on how to support students to develop mathematical explanation that meets the above four criteria.

Having this disciplinary approach at the outset, the purpose of this dissertation is to conceptualize the work of teaching entailed in supporting students to develop mathematical explanation, and particularly, the ways of using the two key instructional resources of discourse resources and collective resources to that end. More specifically, the research questions that guide this dissertation study are:

1. What are the core tasks of teaching to support students to develop mathematical explanation?
2. How are instructional resources used to support these core tasks of teaching?
    a. How are discourse resources used to support these core tasks of teaching?
    b. How are collective resources used to support these core tasks of teaching?

To provide an empirical basis for an analytical-conceptual method, I analyzed the data from the Elementary Mathematics Laboratory (EML), a two-week summer mathematics program for entering fifth graders taught by Professor Deborah Ball at the University of Michigan's School of Education, across multiple years. Among the plethora of records of practices of the EML data documented, I purposefully selected the four mathematical tasks that highlight different features of mathematical explanation: (1)

the brown rectangle problem; (2) the blue and green rectangle problem: (3) the two-coin problem; and (4) the three-permutation problem.

A central premise of this dissertation study is that teaching involves managing dynamic instructional interactions between a teacher and students around the content (Cohen et al., 2003; Lampert, 2001; Cohen, 2011). This premise has permeated into the theoretical, methodological, analytical, and conceptual framework of this dissertation study. One of the unique features of this dissertation is analyzing instructional interactions managed by the same teacher teaching the same mathematical tasks to different cohorts of students sampled from the same school district over multiple years. By holding other variables such as a teacher and content relatively constant but only varying students, this dissertation is designed to untangle the ways in which the same teacher adjusts instruction to meet the students' needs in developing mathematical explanation wherein each cohort of students brings different mathematical ideas, stance, issues, language, ambiguity, and difficulties in explaining the same mathematical tasks.

As an individual case analysis, Chapters 4 through 7 presented the extensive detailed analysis of instructional interactions managed by the same teacher for teaching the same mathematical tasks to different cohorts of students. The individual case analysis identified difficulties that each cohort had in developing mathematical explanation for each mathematical task and created initial characterizations of mathematical supports provided to students to develop mathematical explanation for each mathematical task.

Building on these individual case analyses, the first part of Chapter 8, as a cross-year analysis, discusses the relationship between instructional features and supporting students to develop mathematical explanation. The similarities across multiple years became strong candidates to be scaled up into the coherent structure of supporting students to develop mathematical explanation, whereas the differences across multiple years offered analytical opportunities to examine whether or not a particular instructional feature plays a role in supporting students to develop mathematical explanation. In comparing differences across years, however, this dissertation did not argue about the most effective approach that teachers should adopt to support students to develop mathematical explanation. Instead, for the variables that a teacher usually cannot control such as instructional contexts (e.g., when the mathematical task is taught) and students (e.g., answers that students produced in a private space and answers that students proposed in a public space; the proportion of correct answer to the incorrect answers; the intensity of counterarguments; when the key idea emerges; and mathematical issues that matter the most), I provided arguments whether they might impose different demands to support students to develop mathematical explanation. On the other hand, for the variables that a teacher usually can control such as pedagogical approaches (e.g., selecting; sequencing; making public about what is monitored; and organizing independent work) and mathematical approaches (e.g., using concrete materials; using real-life contexts; and instructional sequence of mathematical tasks), I discussed how a particular approach might interact with the collective construction of mathematical explanation.

The second part of Chapter 8, as a cross-mathematical-task analysis, argues for a distinction between "mathematical-task generality" and "mathematical-task specificity" of constructing mathematical explanation. In constructing mathematical explanation, there are some general issues of explaining in the individual level such as vagueness, lack of clarity, stammering, redundancy, and repetition of explanation and some general issues of explaining in the collective level regarding to hearing others' explanation in an attentive way and expressing disagreement with others' explanation in a respectful way. Beyond these general issues applied to all mathematical tasks, the struggles that students have in developing mathematical explanation do not remain the same across mathematical tasks. These different struggles faced by students are anchored in different mathematical reasoning, warrant, logic, and structure embedded in different mathematical tasks. Whereas the ideas around leading a whole-group discussion and creating a mathematical discourse community are more generic but are not much differentiated by mathematical tasks in the literature, this dissertation argues that mathematical tasks function as a prominent variable in shaping the level of mathematical supports and the role of instructional resources. The four mathematical tasks analyzed in this dissertation study impose different demands in constructing mathematical explanation for the following domains: (1) the level of threshold for accessing the key mathematical ideas; (2) the crucial role of the key incorrect answers; (3) the demand of using accurate language; (4) the demand of using representations; and (5) the demand of organizing solution spaces. Correspondingly, these different demands embedded in mathematical tasks shape the different levels of mathematical supports provided by a teacher in the following domains: (1) scaffolding during set-up and independent work; (2) preserving the key incorrect answers; (3) surfacing language to articulate; (4) connecting representations to articulate; and (5) having a control of using a public space.

Building on these analytical grounds, Chapter 9 introduces a conceptual framework for the work of teaching entailed in supporting students to develop mathematical explanation around four core tasks of teaching—(1) attending to the organic structure of mathematical task; (2) mapping the scope of students' answers onto the targeted mathematical ideas; (3) hearing the mathematical needs embedded in students' explanation; and (4) distributing and building a mathematical talk collectively— and two instructional resources—(1) discourse resources and (2) collective resources. The initial characterizations, categories, and themes emerged from individual-year analysis, cross-year analysis, and cross-mathematical-task analysis are restructured to accommodate the following four conceptual underpinnings that serve as the rationale for decomposing the work of teaching in this dissertation.

First, all of the four core tasks of teaching are requisite components to accomplish the goal of supporting students to develop mathematical explanation, thus missing or malfunctioning one of the four core tasks of teaching makes it difficult to attain its goal. The core tasks of teaching can be thought as distributed activities (c.f., Sleep, 2009), but it might require additional discernment for teachers to figure out whether a particular core task of teaching is necessary for a particular instance of teaching. Second, the commonalities identified across mathematical tasks are taken into account to name core tasks of teaching but the differences observed across mathematical tasks are considered as instructional resources instead. The four core tasks of teaching are equally weighted regardless of mathematics tasks, but the two key instructional resources are differently deployed and loaded across mathematical tasks. Third, beyond the benefits of specifying the granularity, the decomposed practices need to serve to address the targeted problem of teaching that aims to be resolved. Considering that one of the demanding aspects of teaching is the variability, uncertainty, and unexpectedness brought by students, this dissertation decomposes the work of teaching in a way to deal with this issue. The four core tasks of teaching decomposed in this dissertation are structured to gradually increase the demands and complexity of unpacking students' ideas along the student-content arrow in the instructional triangle so that it is more understandable, doable, and learnable for teachers. Lastly, the further decomposition of each core task is formulated as problematic issues that hinder the enactment of each core task of teaching rather than skillful management. The formulation of skillful management can be easily translated to particular actions needed to be taken, but it might over-emphasize particular strategies employed in the work, overlook other possible strategies serve for the work, or misrepresent the nature of the work.

In the following sections, I demarcate my dissertation to prevent possible misinterpretation of findings, discuss the implications for research on teaching—particularly for research design and for decomposition—, and discuss the possible uses of my dissertation for teacher education or professional development programs.

10.2. Demarcating This Dissertation: What It Is and What It Is Not

Due to the analysis of instructional interactions managed by a well-known expert teacher, one might consider that this dissertation is a descriptive study about one expert teacher's routines, improvisations, actions, or decisions. Because of this, one might interpret that the findings of this dissertation study set high standards for novice teachers that are not feasible for them to implement, thus it is not generalizable or applicable for other teachers. To prevent some of these potential misinterpretations, this section demarcates this dissertation: what this dissertation is and what this dissertation is not. I briefly discussed some cautions with interpreting this dissertation in Chapter 3.3.3, but this section further articulates the positions taken in this dissertation in terms of (1) the main target of study; (2) the use of longitudinal data for the analysis; (3) the use of lesson plan for the analysis; (4) the focal aspect of the practice of explanation; and (5) the interpretations of findings. Table 10.1 provides a summary of demarcating this dissertation study.

Table 10.1. Demarcating this dissertation

	What This Dissertation Is	What This Dissertation Is Not
(1) The main target of study	• Identifying the entailments in the work of teaching to support students' mathematical learning • Identifying the demands imposed by students and the adjustment of instruction to meet students' needs • Identifying mathematical demands entailed in the work of teaching	• Describing an expert teacher's belief, motivation, or intention which leads to a particular decision • Describing an expert teacher's routines, in-the-moment decisions, or improvisations • Evaluating the quality of instruction or the level of MKT
(2) The use of longitudinal data for the analysis	• Using the longitudinal data as multiple cases of instructional interactions managed • Using the longitudinal data to examine how different approaches might differently leverage the construction of mathematical explanation	• Using the longitudinal data chronologically to reveal teacher learning over time • Using the longitudinal data to nominate the most effective approach that results in the richest mathematical explanation
(3) The use of lesson plan for the analysis	• Serving as resources to elucidate instructional goals, lesson structure, mathematical territory, and anticipations	• Evaluating the quality of planning • Evaluating the degree of fidelity to the planned lessons
(4) The focal aspect of the practice of explanation	• Analyzing how an individual student's idea mathematically expands through a collective work • Examining how mathematical explanation is collectively constructed in a more disciplinary robust, profound, and meaningful way	• Quantifying the amount, length, duration, frequency, type, and distribution of discursive activities • Evaluating whether each statement made by an individual student can be counted or acceptable as a mathematical explanation
(5) The interpretation of findings	• Serving as analytical and conceptual tools to parse out the complexity in the work of teaching so that it is understandable, doable, and learnable	• Prescribing strategies or techniques that teachers should follow in step-by-step

First, this dissertation is studying teaching rather than studying a teacher. Certainly, it is not easy to make a distinction between studying teaching and studying a teacher because the work (i.e., teaching) is done by people (i.e., teacher) who engage in the work with taking responsibility to achieve its goals. The findings of this dissertation are neither to describe what one expert teacher simply does, says, or knows nor to evaluate the teacher's instructional quality or the teacher's level of MKT. Throughout the dissertation, even not exhaustively, I often made a deliberate effort to write the subject of sentence as the entailments in the work rather than the teacher. For instance, instead of saying "Ms. Ball asked not to agree or disagree," I translated it as "legitimizing the disagreement about answers allows preserving the incorrect answers." More importantly, this dissertation identifies the entailments in the work of teaching to support students' mathematical learning, the demands imposed by students and the adjustment of instruction to meet students' needs, and mathematical demands entailed in the work of teaching.

Second, due to the longitudinal data collected consecutively, one might be interested in what the teacher learn over time through the reflection on the experiences of managing instructional interactions or what motivates such instructional changes over time. As I discussed in Chapter 3, this dissertation does not chronologically track the changes over time but treat them as multiple cases of instructional interactions managed. Thus, the reflections that the teacher made and why the teacher made such changes are beyond the scope of this dissertation study. In addition, because of the multiple cases, one might easy to nominate the most effective approach that results in the richest mathematical explanation, but this dissertation examines how different approaches might differently leverage the construction of mathematical explanation.

Third, a well-designed lesson plan serves for teachers to organize the instruction effectively, to steer instruction to achieve goals, to prevent the instruction veer off the key mathematical points (Sleep, 2009), and to prepare for the students' responses, but this dissertation does not evaluate the quality of lesson plan or the fidelity to lesson plan. Instead, this dissertation uses the lesson plan as resources to elucidate instructional goals, lesson structure, mathematical territory, and anticipations. Given that the mathematical tasks are not from the widely used curriculum materials but devised by the teacher and the group of researchers in the EML program, the lesson plans provide rich contextual information about mathematical tasks and students.

Fourth, the focal aspect of the practice of explanation is the development of explanation over time through the collective work in a disciplinary meaningful way. Thus, this dissertation does not quantify the amount, length, duration, frequency, type, and distribution of talk between the teacher and students, but analyzes how an individual student's mathematical ideas expand through a collective work. In addition, this dissertation does not evaluate whether each statement made by an individual student can be counted or acceptable as a mathematical explanation but examine how mathematical explanation is collectively constructed in a more disciplinary robust way.

Lastly, the findings of this dissertation should not be interpreted as prescribing strategies or techniques that teachers should follow in a step-by-step manner, but should be interpreted in a way of serving as analytical and conceptual tools to parse out the complexity of teaching so that it is understandable, doable, and learnable.

10.3. Implications for Research on Teaching

A main product of this dissertation is the conceptual framework for the work of teaching entailed in supporting students to develop mathematical explanation—decomposing the work into four core tasks of teaching and two instructional resources—as well as identifying mathematical task as a prominent variable that shapes the level of mathematical supports and the role of instructional resources. Besides these findings, which provide useful practical ideas for teachers and teacher educators, this dissertation has important implications for research on teaching. In the following sub-sections, I discuss the ways in which this dissertation advances a research design that particularly matters for studying teaching and that contributes to establishing a grammar of decomposition.

10.3.1. Research Design Serving to Address The Targeted Problem of Teaching

In studying teaching and learning mathematics, researchers often use multiple cases of teaching (e.g., lessons taught by multiple teachers) to understand the nature of teaching or to characterize teachers who have certain knowledge, beliefs, and approaches. Some researchers do not further distinguish multiple cases from one another, but, instead, they synthesize multiple cases to capture a wide range of phenomenon. Others deliberately select multiple contrasting cases, in which one group usually performs better than the other, such as novice teachers versus expert teachers, less knowledgeable teachers versus more knowledgeable teachers, and teachers who have low fidelity to curriculum materials versus teachers who have high fidelity to curriculum materials. Unlike an experimental study that controls for variables in order to isolate effects of those of primary interest, an observational study often does not have much control of confounding variables. For this reason, the findings from these studies are likely to suffer from validity threats (Shadish, Cook, & Campbell, 2002). For example, suppose that a researcher recruits two teachers with different levels of teacher knowledge to characterize instructional quality and learning opportunities for students through a careful selection process: that is, high knowledgeable first grader teacher and less knowledgeable fifth grader teacher based on the scores gained from a valid and reliable instrument that measures knowledge needed for teaching. Even if the instruction managed by the highly knowledgeable teacher is contrasted to the instruction managed by the less knowledgeable teacher, the differences in instructional interactions might be explained by grade-level differences (e.g., students' dispositions and motivations; content) rather than being solely attributable to the difference in teacher knowledge. As teachers change, other key agents of the instructional triangle (students and content) change as well, thus research design needs to isolate key variables influencing instructional interactions. In this dissertation, I argue that research design needs to leverage the targeted problem of teaching and the findings from such studies need to be carefully interpreted with regard to the degree of control of the key variables of instruction.

The unique research design adopted in this dissertation is somewhat rare but it is not a totally new one. Referring to their "same teacher-different classes" research program, Even and her colleagues (e.g., Even and Kvatinsky; 2009; Even and Kvatinsky, 2010; Eisenmann and Even, 2009; Eisenmann and Even, 2011) have drawn attention to the interplay between curriculum, teacher (e.g., beliefs), and students (e.g., level of performance) and have examined whether the interplay among these factors shape different learning opportunities for students in more constrained conditions. Even (2014) writes:

> The unique methodology of the research program *Same Teacher—Different Classes*, which examines teaching and learning mathematics in different classes with the same teacher and with different teachers, enabled us to carefully examine interactions among curricula, teachers, and classes that are not easily detectable, and it revealed the complex ways in which they shape students' opportunities to learn mathematics. As this chapter suggests, attending to the interplay among teachers, the curriculum, and classes has great potential to contribute to sophisticated understanding of teaching and learning in the classroom in general and to curriculum enactment in particular. (p.472)

By holding the variables of instructional interactions somewhat constant, a line of research using "the same teacher-different classes" method has potential to contribute to an understanding different learning opportunities provided students in a more subtle way. The main differences between Even's work and the approach taken in my dissertation are as follows. First, I put more rigorous constraints on the variations across cases: the similar composition of students rather than different composition of students across cases. Second, I did not focus on the teacher's particular approach, beliefs, or curriculum fidelity but focused on the entailments and demands of teaching. Third, the differences across cases serve as analytical opportunities to investigate the entailments of teaching rather than the approach that the teacher has taken for the particular characteristics of students. The remainder of this section lays out three research designs that hold some of the key variables of instruction constant and discusses potential benefits of using such a research design in conceptualizing teaching.

The first research design is holding a teacher and content constant but varying students as adopted in this dissertation study. Because of the context of elementary teaching, my dissertation analyzes the instructional interactions managed by the same teacher over years, but the data, as studied here, are not longitudinal in nature. In a secondary setting, data drawn from one teacher teaching the same mathematical content across multiple classes, with students deemed to be sampled from the same population, might also serve in a similar way. This research design addresses the issue of the uncertainty of teaching brought by students. It can be used to illuminate ways in which instruction unfolds differently depending on ideas that students bring and how instruction might be adjusted to meet students' needs.

Another research design is holding a teacher and students but varying the content. I did not primarily compare how the instructional interactions managed by the same teacher with the same students unfold differently across mathematical tasks (e.g., the brown rectangle problem in the EML 2010 vs. the two-coin problem in the EML 2010) but provided more general characteristics across mathematical tasks. Given that this dissertation argues that mathematical task shapes the level of mathematical supports and the role of instructional resources, only varying the mathematical tasks while controlling a teacher, students, and instructional contexts reveals the mathematical-task specificity issues. One caution would be to hold the temporal dimension relatively constant, either sampled the mathematical tasks taught at the beginning of an academic year or sampled the mathematical tasks taught at the end of an academic year.

Lastly, varying only one variable at the time is the ideal but it might not be always feasible because of the time, cost, and availability. In such cases, controlling the content is beneficial because it illuminates the various ways of implementing the same mathematical tasks. However, it needs to consider different levels of fidelity to the content being taught, different levels of teacher knowledge, different levels of instructional quality, and different populations of students. These research designs, holding the key elements of instructional interations somewhat constant, are well worth further attention and development.

10.3.2. Decomposition of Teaching Beyond Specifying Its Granularity

In the studying of teaching practices across three different professions (clergy, clinical psychology, and teaching), Grossman et al. (2009) identify decomposition of practice, along with representations of practice and approximations of practice, as the key for understanding the pedagogies of practice in professional education. Responding to this call, several researchers have recently begun to decompose the complex practice of teaching into its constituent components (e.g., Boerst et al., 2011; Sleep, 2009). Addressing the need for developing a common language to describe and analyze teaching, Grossman and McDonald (2008) write:

> Such a framework for teaching would require a careful parsing of the domain, an effort to identify the underlying grammar of practice, and the development of a common language for naming its constituent parts. […] This effort to parse teaching would need to respect the difficulty of breaking apart such a complex system of activity and the dangers of doing irreparable harm to the integrity of the whole by making incisions at the wrong places. Such a framework could inform both research on teaching and the improvement of professional education (p.186).

Despite the shared interests in decomposition, the grammar of decomposition—the structure of decomposition, the language to describe the decomposed practice, and the granularity of the decomposed practice—has not been well established, articulated, and discussed yet in research on teaching and teacher education. For this reason, it is often unclear what counts as a decomposition of practice versus what is just a collection of approaches, strategies, or techniques. Recognizing a lack of clarification about the method of decomposition in studying teaching, I made an attempt to elaborate the ideas underlying the decomposition of the work of teaching entailed in supporting students to develop mathematical explanation in Chapter 9.

Extending the issue of how to decompose the complex practice for a particular domain of teaching, this section discusses some functional aspects of decomposition in a much broader context beyond the benefits of specifying its granularity. In doing so, I first began by making a reference to the philosophical non-fiction, titled "*Zen and the Art of Motorcycle Maintenance*" written by Robert M. Pirsig (1974), which casts the meaning of quality through an illustration of motorcycle maintenance. During a motorcycle road trip from Minnesota to Northern California by the author and his son, accompanied by his close friends John and Sylvia, they discussed many philosophical ideas, especially searching for the concept of quality.

Throughout the book, the author describes two approaches to life: a romantic quality approach and a classic quality approach. On a romantic quality side, his friend, John, having an expensive new motorcycle, did not learn how to maintain his motorcycle, just hoping nothing would happen to it. When a problem occurred with the motorcycle, he became frustrated. On the other hand, the narrator, an owner of an old motorcycle, having knowledge of the subdivided components of his motorcycle, predicted, diagnosed, and repaired the problem. The decomposition of a motorcycle (see Figure 10.1) serves as a useful tool for him to comprehend the mechanism of a motorcycle so that a motorcycle is well maintained without sacrificing expensive costs and further creating fatal damages caused by the late detection of problems.

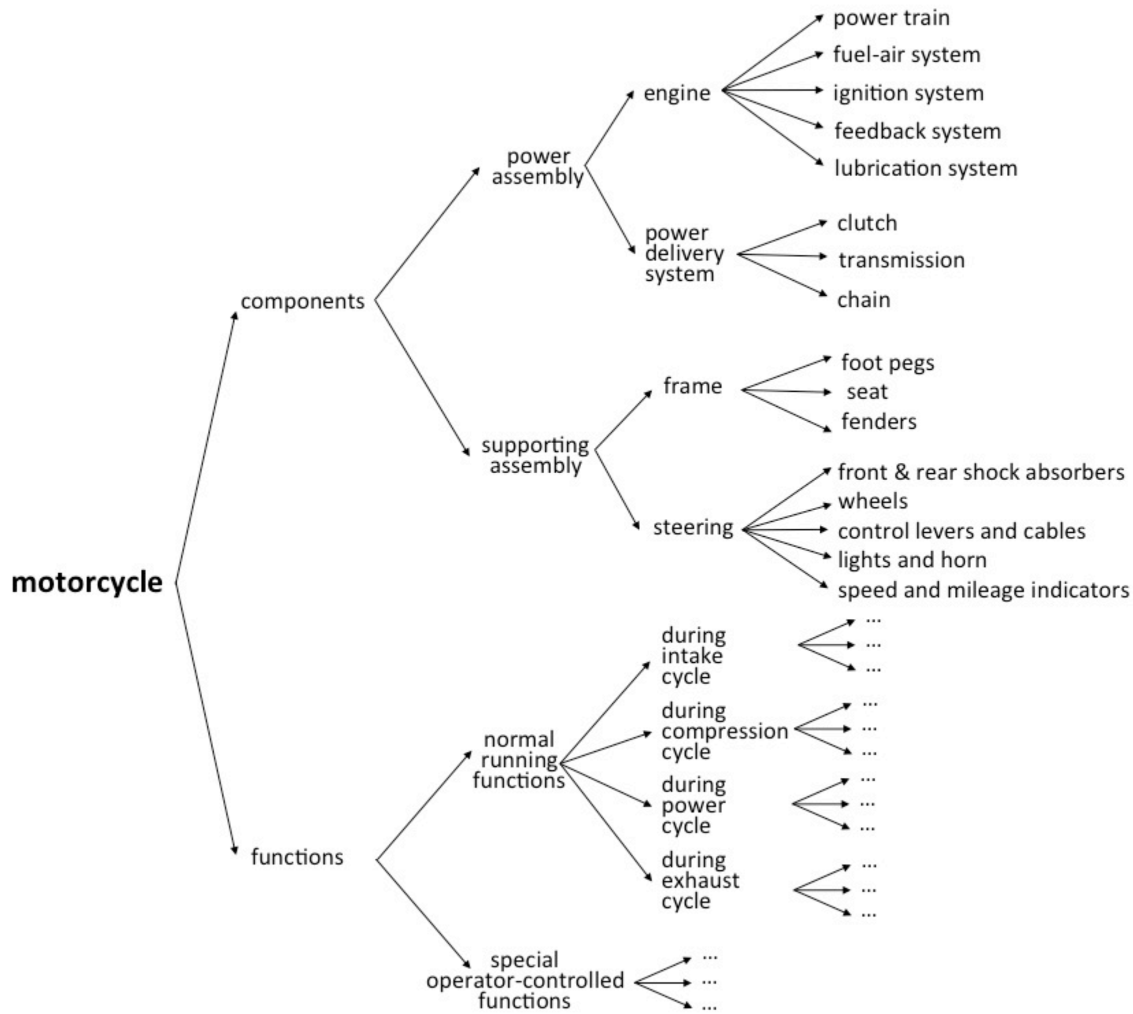

Figure 10.1. A decomposition of motorcycle (Pirsig, 1975, pp. 73-74, a diagram constructed based on the texts)

As illustrated in Figure 10.1, the author decomposes a motorcycle into four structural levels with fine-grained details. Relating to several analytical questions I raised for decomposing the work of teaching in Chapter 3, the following two interesting observations can be made. First, the grain size within the same hierarchical level is quite similar, having shared names such as "… assembly" under the component branch or "during … cycle" under the normal running functions branch. This provides insights about the relationship between the subdivided components within the same hierarchical level in decomposing the work of teaching. Second, the feature of subdivided components in the final level under the components branch is the physical motorcycle parts that are visible thus repairable or replaceable. As the author writes "with a single stroke of analytic thought he split the whole world into parts of his own choosing, split the parts and split the fragments of the parts, finer and finer and finer until he had reduced it to what he wanted it to be." (p.76); the final unit of decomposition needs to match with the underlying goal of decomposition and how the decomposition can be used to predict, diagnose, and resolve the issues.

Admittedly, the decomposition of mechanical object (i.e., motorcycle) is quite different from the decomposition of an enacted practice (i.e., teaching), but the contemplation of how the motorcycle is decomposed for the purpose of maintenance by the author casts light on several important issues that need to be considered when decomposing the work of teaching.

First, the decomposed components are not a list of step-by-step instructions to follow but can be used as a reference to predict, diagnose, and resolve problems by paying continual attention to those constituent components. Like John, having a romantic quality approach, teachers also might expect that students bring up correct answers with clear, accurate, consistent, well-organized, logical, and legitimate reasons to explain solutions to mathematical tasks, instead of having knowledge of the subcomponents of the work to support students to develop mathematical explanation and preparing for it.

Second, the linkage between the decomposed components needs to be explicitly addressed. In making the linkage between different hierarchical levels, in the example of motorcycle decomposition illustrated above, the author operationalizes the "components" branch using the verb "contains" whereas he operationalizes the "functions" branch using the verb "cause." Likewise, the operators across multiple hierarchical levels (how to define the relationship between "the first-level decomposition" and "the second-level decomposition") and the operators within the same hierarchical level (how the decomposed practices within the same hierarchical level are related to each other) need to be more explicitly specified.

Lastly, we need to pay attention to how the knife is used in the decomposition instead of indiscriminately adopting the products of decomposition. Unlike how the motorcycle is decomposed for the purpose of maintenance, an instructor for the motorcycle driving school might use the knife differently that serves well to the goal of driving a motorcycle safely and of preventing possible motorcycle accidents. Taking a metaphor, an onion can be cut into different ways such as slice, dice, chop, or ring that match with the purpose of cooking. There is not one single way of cutting an onion, but the way of cutting an onion matches with the food, harmony with other ingredients, and its functions. Considering that one of the demanding aspects of teaching is the variability, uncertainty, and unexpectedness brought by students, this dissertation decomposes the work of teaching in a way to deal with this issue. The four core tasks of teaching decomposed in this dissertation are structured to gradually increase the demands and complexity of unpacking students' ideas along the student-content arrow in the instructional triangle so that it is more doable for teachers. Beyond utilizing the product of decomposition, specifying the structure and underlying rationale of decomposition greatly contribute to establish a grammar of decomposition in research on teaching and teacher education.

10.4. Implications for Teacher Education

The two fields of research, research on teaching and research on teacher education, are closely interrelated; thus its boundary is not quite clear. In recent years, with increased interest in using records of practice and teaching the core practices in teaching teachers—both for preservice teachers and inservice teachers—research on teaching has been often directly translated to the curriculum of teacher education. For instance, the five practices for orchestrating a productive whole-group discussion proposed by Stein et al. (2008)—anticipating, monitoring, selecting, sequencing, and connecting—has been readily translated and directly used for teacher education (e.g., Tyminiski, Zambak, Drake, & Land, 2014). Basically, in a similar vein, the conceptual framework of this dissertation—four core tasks of teaching and two instructional resources—can be used as a curriculum of teacher education. This section lays out the potential uses of this dissertation in the contexts of teacher education and addresses the issues that can be raised. Using a Q & A format, I offer my responses to hypothetical questions that might be raised by teacher educators or professional development practitioners who are interested in using the findings of my dissertation in the following areas: (1) eligibility and target audience; (2) appropriateness, availability, and use of records of practice; (3) designing mini-courses or modules; and (4) implementing mini-courses or modules.

Eligibility and Target Audience

Q1: Your conceptual framework is developed through the analysis of instructional interactions managed by one expert teacher. Is the target audience who would primarily benefit from using this conceptual framework experienced or advanced teachers (e.g., National Board Certified teachers or math coaches)? Is the conceptual framework also applicable for preservice teachers who do not have any field experiences or for novice teachers who do not have extensive experiences yet? If so, how might the use of the conceptual framework be different for experienced teacher and for novice teachers?

A1: Even though the conceptual framework is developed through an iterative data analysis of instructional interactions managed by an expert teacher, the use of conceptual framework is not restricted or limited to experienced or advanced teachers. In conversations with other colleagues, both formally and informally, I often realize that some do not have trouble with generalizing findings from novice teachers but raise the issue of generalizability of findings drawn from the data of expert teachers. As I addressed throughout my dissertation, the conceptual framework is not a description of what one expert teacher simply does, says, or decides, but it is about the entailment in the work of teaching to achieve the goal of helping student learn mathematics. Thus, the conceptual framework is not limited to experienced or advance teachers. Given that novice teachers or preservice teachers do not have sufficient exposure to the management of instructional interactions—from the perspective of a teacher, not from the perspective of their apprenticeship of observation as students—one might customize the use of the conceptual framework according to the targeted audience. One idea might be that the conceptual framework can be introduced with more rich and frequent use of records of practices for novice teachers or preservice teachers than experienced teachers. Another idea might be, considering that the four core tasks of teaching increase the demand of unpacking students' mathematical ideas, one might mainly focus on the first two core tasks of teaching—attending to the organic structure of mathematical tasks and mapping the scope of students' answers onto the targeted mathematical ideas—rather than the last two core tasks of teaching—hearing the mathematical needs embedded in students' mathematical ideas and distributing and building a mathematical talk collectively. This would be similar to the approach taken by Morris, Hiebert, and Spitzer (2009), which mainly focuses on "unpacking learning goals" as a prime candidate for teaching preservice teachers because of its more modest demand on managing dynamic interactions with students in real-time contexts.

Q2: Your conceptual framework is grounded on the empirical data drawn from elementary mathematics classrooms. Is the conceptual framework also applicable for secondary mathematics classroom settings?

A2: As pointed out, the conceptual framework of this dissertation study is grounded on the empirical data drawn from elementary mathematics classrooms, but not tested in the secondary mathematics classrooms yet. Based on my experiences of teaching and studying teaching in a variety of contexts, I expect that this conceptual framework would be generally applicable for Grades K-8, but might not be entirely applicable for Grades 9-12. One thing to consider is that this conceptual framework is for supporting students to develop mathematical explanation rather than general instructional discourse patterns or leading whole-group discussions. Given that mathematical tasks in Grades 9-12 require more formal forms of proof, the conceptual framework of this dissertation might not be entirely applicable for such contexts.

Availability, Appropriateness, and Use of Records of Practice

Q3: Should I use videos for teaching the conceptual framework to teachers? Instead, can I just use a mathematical task to discuss the four core tasks of teaching and two instructional resources without analyzing actual records of practice? If I do not have rich records of practice to use for teaching teachers, what might be possible options?

A3: Due to the practical-based approach, it would not be easy for teachers to generate ideas for the problematic issues that hinder the enactment of each core task of teaching and the role of instructional resources for a particular mathematical task that they would like to teach without using records of practice. Using the extensive detailed analysis provided in this dissertation, you can discuss the core tasks of teaching and instructional resources for the given mathematical task. If teachers are more familiar with the details of the conceptual framework, one can discuss the first two core tasks of teaching without records of practice because those core tasks of teaching emphasize the content side and rely less on students. After teaching the mathematical task in their own classroom, one can discuss the last two core tasks of teaching.

Q4: If I would like to use records of practice for teaching this conceptual framework for teachers, should I use records of practice from the same teacher teaching the same mathematical tasks to different cohorts of students?

A4: No. The conceptual framework is developed through data of the same teacher teaching the same mathematical tasks to multiple cohorts of students, but the conceptual framework can be taught using a single record of practice.

Q5: If I would like to use records of practice for teaching this conceptual framework for teachers, is it better to use records of practice from an expert teacher or from a novice teacher who has struggles with teaching?

A5: Records of practice from either an expert teacher or a novice teacher can provide insight into practice. Whatever records of practice are used, it is important to remind teachers that it is not to describe or to evaluate what the teacher in the video does or says but to pay attention to the entailments of teaching.

Q6: The mathematical tasks analyzed in your conceptual framework are not from commonly used curriculum materials, but they are very purposefully designed by the research group at the EML. In using your conceptual framework with teachers, can I use mathematical tasks from curriculum materials that are not highly cognitively demanding?

A6: The conceptual framework is not restricted to highly cognitively demanding mathematical tasks or to mathematical tasks with multiple answers. Even for mathematical tasks with only one answer and focusing on an algorithm, how to support students to develop mathematical explanation can be discussed. If the mathematical tasks from the curriculum materials are not rich in developing mathematical explanation, the extensive discussion about the first core task of teaching—attending to the organic structure of mathematical task—would be beneficial.

Designing Mini-Courses or Modules

Q7: If I would like to teach this conceptual framework in my professional development programs, what might be the appropriate credit units? How much time does it take to teach this conceptual framework?

A7: It depends on the pace of teaching this conceptual framework. However, it is important to remember that this conceptual framework needs to be addressed at least for two mathematical tasks which have different nature of mathematical explanation in order to explicitly address the idea that mathematical task shapes the level of mathematical supports and the role of instructional resources that I addressed in the second part of Chapter 8.

Q8: Is there any instructional sequence that I need to consider? Do I need to teach "four core tasks of teaching" before "two instructional resources"? Or do I need to teach "two instructional resources" before "four core tasks of teaching"? Within the four core tasks of teaching, is there any specific instructional sequence to consider for teaching?

A8: Given that instructional resources serve for the work of teaching, it would be good to teach "four core tasks of teaching" before "two instructional resources." In addition, the four core tasks of teaching are not necessarily chronologically or sequentially ordered, but a core task of teaching in the less complex side (e.g., attending to the organic structure of mathematical task) might be taught prior to a core task of teaching in the more complex side (e.g., distributing and building mathematical talk collectively) because of the less complexity in unpacking students' ideas.

<u>Implementing Mini-courses or Modules</u>

Q9: What activities should be preceded before watching the video to analyze the work of teaching entailed in supporting students to develop mathematical explanation? What activities should be followed up after watching the video? What guiding questions can I use while watching the video?

A9: Given that mathematical tasks shape the level of mathematical supports and the role of instructional resources to serve the work, it is important to do the mathematical task analysis before introducing the conceptual framework to preservice teachers or inservice teachers. This includes drafting mathematical explanations for the given mathematical task that meet the four criteria introduced in the earlier section, anticipating students' initial explanations that are likely to be proposed (not just a complete mathematical explanation, but possibly ambiguous, inaccurate, incoherent, or insufficient explanations), and doing the mathematical analysis of the actual records from students' written explanations and verbal explanations. Through these activities, preservice teachers or inservice teachers will have opportunities to see that students' mathematical explanations are anchored in the nature of mathematical tasks, thus both "mathematical-task-generality" and "mathematical-task-specificity" approaches are necessary.

Q10: At the end of mini-courses or modules, how can I assess the participating teachers' performances in supporting students to develop a mathematical explanation?

A10: My dissertation mainly addresses the necessary components of supporting students to develop mathematical explanation rather than how to teach those practices in a repeated manner while providing feedback on learning those practices. The conceptual framework is an analytical and conceptual tool for teachers to use, but is not fully served as an evaluation criteria yet.

I would like to end this dissertation with some issues that need to be considered before using the findings of this dissertation in teacher education settings. Among three key ideas for teaching practices in professional education, this dissertation mainly conceptualizes the decomposition of practice, parsing out the complex work of teaching into its constituent components by gradually increasing the demands of unpacking students' ideas but does not pay a particular attention to two other important aspects—representations of practice (c.f., Chazan, Herbst, and Sela, 2011), which lay out different ways that the practice is represented, and approximations of practice (c.f., Lampert et al., 2013; Beasley, 2014), which focus on rehearsing these practices in a repeated manner and providing specific feedback on the rehearsal. In this sense, my Q & A section above has some limitations because questions are addressed without a clear sense of the representations and approximations that might shape the use of my dissertation findings.

REFERENCES

Amsler, R. A. (1984). Lexical Knowledge bases. *In Proceedings of COLING-84*. Standford University, Standford, CA, July (pp.458-459).

Balacheff, N. (1988). Aspects of proof in pupils' practice of school mathematics. In D. Pimm (Ed.), *Mathematics, teachers and children* (pp.216-235). London: Holdder & Stoughton.

Balacheff, N. (1999). Is argument an obstacle? Invitation to a debate*. International Newsletter on the Teaching and Learning of Mathematical Proof.*

Ball, D. L. (1993). With an eye on the mathematical horizon: Dilemmas of teaching elementary school mathematics. *The Elementary School Journal, 93*(4), 373-397.

Ball, D. L. (1997). What do students know? Facing challenges of distance, context, and desire in trying to hear children. In *International handbook of teachers and teaching* (pp. 769-818). Springer Netherlands.

Ball, D. L. (2004). What are teachers learning? Invited plenary address to the National Council of Supervisors of Mathematics, Philadelphia, PA.

Ball, D. L. & Forzani, F. M. (2009). The work of teaching and the challenge for teacher education*. Journal of Teacher Education, 60*(5), 497-511.

Ball, D. L. & Hoover, M. (2014). Transforming research to transform mathematics instruction. In Y. Li, E. A. Silver, & S. Li (Eds.), *Transforming mathematics instruction: Multiple approaches and practices*, Advances in Mathematics Education (pp. 549-557). Switzerland: Springer International.

Ball, D. L., & Bass, H. (1999). Making believe: The collective construction of public mathematical knowledge in the elementary classroom. *Yearbook-National Society for the Study of Education*, *1*, 193-224.

Ball, D. L., & Bass, H. (2003). Making mathematics reasonable in school. In J. Kilpatrick, W.G. Martin, and D. Schifter (Eds.), *A Research Companion to Principals and Standards for School Mathematics* (pp.27-44). Reston, VA: National Council of Teachers of Mathematics.

Ball, D. L., Lewis, J., & Thames, M. H. (2008). Chapter 1: Making Mathematics Work in School. *Journal for Research in Mathematics Education. Monograph*, 13-201.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching what makes it special? *Journal of Teacher Education*, *59*(5), 389-407.

Bass, H. (in preparation). Seeking, using, and developing mathematical structure. Unpublished manuscript, University of Michigan, Ann Arbor, MI.

Baumert, J., Kunter, M., Blum, W., Brunner, M., Voss, T., Jordan, A., ... & Tsai, Y. M. (2010). Teachers' mathematical knowledge, cognitive activation in the classroom, and student progress. *American Educational Research Journal*,*47*(1), 133-180.

Beasley, H. L. (2014). *Investigating Opportunities to Develop the Practice of Responding to Students' Contributions to Mathematical Discourse in Embedded Rehearsal*. Unpublished doctoral dissertation, University of Michigan, Ann Arbor, MI.

Boero, P. (1999). Argumentation and mathematical proof: A complex, productive, unavoidable relationship in mathematics and mathematics education. *International Newsletter on the Teaching and Learning of Mathematical Proof*, *7*(8).

Boerst, T., Sleep, L., Ball, D. L., & Bass, H. (2011). Preparing teachers to lead mathematics discussions. *Teachers College Record*, *113*(12), 2844-2877.

Borko, H., & Livingston, C. (1989). Cognition and improvisation: Differences in mathematics instruction by expert and novice teachers. *American educational research journal*, *26*(4), 473-498.

Chapin, S. H., O'Connor, C., & Anderson, N. C. (2009). *Classroom discussions: Using math talk to help students learn, grades K-6*. Math Solutions.

Chazan, D., Herbst, P., & Sela, H. (2011). Instructional alternatives via a virtual setting: Rich media supports for teacher development. In *Constructing Knowledge for Teaching Secondary Mathematics* (pp. 23-37). Springer

Chi, M. T., Bassok, M., Lewis, M. W., Reimann, P., & Glaser, R. (1989). Self-explanations: How students study and use examples in learning to solve problems. *Cognitive Science*, *13*(2), 145-182.

Chi, M. T., Leeuw, N., Chiu, M. H., & LaVancher, C. (1994). Eliciting self‐explanations improves understanding. *Cognitive science*, *18*(3), 439-477.

Cobb, P., Wood, T., Yackel, E., McNeal, B. (1992). Characteristics of classroom mathematics traditions: An interactional analysis. *American Educational Research Journal, 29*(3), 573-604.

Cohen, D. K. (2011). *Teaching and its predicaments*. Cambridge, MA: Harvard University Press.

Cohen, D. K., Raudenbush, S. W. & Ball, D. L. (2003). Resources, instruction, and research. *Educational Evaluation and Policy Analysis, 25*(2), 119-142.

Common Core State Standards Initiative. (2010). Common Core State Standards for Mathematics. Washington, DC: National Governors Association Center for Best Practices and the Council of Chief State School Officers. Retrieved from http://www.corestandards.org/assets/CCSSI_Math%20Standards.pdf

Corbin, J., & Strauss, A. (2008). *Basics of qualitative research: Techniques and procedures for developing grounded theory*. Sage, Thousand Oaks, CA.

Creswell, J. W. (2013). *Research design: Qualitative, quantitative, and mixed methods approaches*. Sage publications.

Doyle, W. (1986a). Classroom organization and management. In M. C. Wittrock (Ed.), *Handbook of research on teaching* (3rd ed., pp.392-431). New York: Macmillan.

Doyle, W. (1986b). Content representation in teachers' definitions of academic work. *Journal of Curriculum Studies, 18*(4), 365-379.

Doyle, W. (1988). Work in mathematics classes: The context of students' thinking during instruction. *Educational Psychologist, 23*(2), 167-180.

Doyle, W., & Cater, K. (1984). Academic tasks in classrooms. *Curriculum Inquiry, 14*(2), 129-149.

Eisenmann, T., & Even, R. (2009). Similarities and differences in the types of algebraic activities in two classes taught by the same teacher. In J. Remillard, B. Herbel-Eisenmann & G. Lloyd (Eds), *Teachers' use of mathematics curriculum materials: Research perspectives on relationships between teachers and curriculum* (pp.152-170). New York, NY: Routledge.

Eisenmann, T., & Even, R. (2011). Enacted types of algebraic activity in different classes taught by the same teacher. *International Journal of Science and Mathematics Education, 9(*4), 867-891.

Ericsson, K. A. (2005). Recent advances in expertise research: A commentary on the contributions to the special issue. *Applied Cognitive Psychology*, *19*(2), 233-241.

Erlwanger, S. H. (1973). Benny's conception of rules and answers in IPI mathematics. *Journal of Children's Mathematical Behavior 1*, 7-26.

Even, R. (2014). The interplay of factors involved in shaping students' opportunities to learn mathematics. In *Transforming Mathematics Instruction* (pp. 459-474). Springer International Publishing.

Even, R., & Kvatinsky, T. (2009). Approaches to teaching mathematics in lower-achieving classes. *International Journal of Science and Mathematics Education, 7*(5), 957–985.

Even, R., & Kvatinsky, T. (2010). What mathematics do teachers with contrasting teaching approaches address in probability lessons? *Educational Studies in Mathematics, 74*(3), 207–222.

Fischbein, E., & Gazit, A. (1988). The combinatorial solving capacity in children and adolescents. *Zentralblatt fuer Didaktitk der Mathematik, 5*, 193–198.

Forman, E, A. & Larreamendy-Joerns, J. (1998). Making explicit the implicit: Classroom explanation and conversational implicatures. *Mind, Culture, and Activity, 5*(2), 105-113.

Fravillig, J. (2001). Strategies for advancing children's mathematical thinking. *Teaching Children Mathematics, 7*, 454-459.

Gee, J. P. (2014). *An introduction to discourse analysis: Theory and method*. Routledge.

Ghousseini, H. (2008). *Learning with routines: Preservice teachers learning to lead classroom mathematics discussions*. Unpublished doctoral dissertation, University of Michigan, Ann Arbor, MI.

Grossman, P. (2011). Framework for teaching practice: A brief history of an idea. *Teachers College Record*, *113*(12), 2836-2843.

Grossman, P., & McDonald, M. (2008). Back to the future: Directions for research in teaching and teacher education. *American Educational Research Journal*, *45*(1), 184-205.

Grossman, P., Compton, C., Igra, D., Ronfeldt, M., Shahan, E., & Williamson, P. (2009). Teaching practice: A cross-professional perspective. *The Teachers College Record*, *111*(9), 2055-2100.

Hafner, J., & Mancosu, P. (2005). The varieties of mathematical explanation. In *Visualization, Explanation and Reasoning Styles in Mathematics* (pp. 215-250). Springer Netherlands.

Heaton, R. (2000). *Teaching Mathematics to the new standards: Relearning the dance*. New York: Teachers College Press.

Henningsen, M., & Stein, M. K. (1997). Mathematical tasks and student cognition: Classroom-based factors that support and inhibit high-level mathematical thinking and reasoning. *Journal of Research in Mathematics Education, 28*(5), 524-549.

Herbel-Eisenmann, B. A., & Otten, S. (2011). Mapping mathematics in classroom discourse. *Journal for Research in Mathematics Education, 42*(5), 451-485.

Herbel-Eisenmann, B.A., Drake, C., & Cirillo, M. (2009). "Mudding the clear waters": Teachers' take-up of the linguistic idea of revoicing. *Teaching and Teacher Education, 25*(2), 268-277.

Hersh, R. (1993). Proving is convincing and explaining. *Educational Studies in Mathematics*, *24*(4), 389-399.

Hiebert, J. (2003). *Teaching mathematics in seven countries: Results from the TIMSS 1999 video study*. DIANE Publishing.

Hiebert, J., & Grouws, D. A. (2007). The effects of classroom mathematics teaching on students' learning. *Second handbook of research on mathematics teaching and learning*, *1*, 371-404.

Hiebert, J., Gallimore, R., & Stigler, J.W. (2002). A knowledge base for the teaching profession: What would it look like and how can we get one? *Educational Researcher*, *31*(5), 3-15.

Hill, H. C., Ball, D. L., & Schilling, S. G. (2008). Unpacking pedagogical content knowledge: Conceptualizing and measuring teachers' topic-specific knowledge of students. *Journal for Research in Mathematics Education*, *39*(4), 372-400.

Hill, H. C., Rowan, B., & Ball, D. L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal, 42*(2), 371-406.

Hunter, M. (1979). "Teaching is decision making" *Educational Leadership 37*(5), pp.379-385.

Johnson, D. W., & Johnson, R. T. (1975). *Learning together and alone: Cooperation, competition, and individualization*. Englewood Clifffs, N. J.: Precentice-Hall.

Kerlinger, F. N. (1979). *Behavioral research: A conceptual approach*. Harcourt School.

Knuth, E. J. (2002a). Secondary school mathematics teachers' conceptions of proof. *Journal for Research in Mathematics Education, 33*(5), 379-405.

Knuth, E. J. (2002b). Teachers' conceptions of proof in the context of secondary school mathematics. *Journal of Mathematics Teacher Education*, *5*(1), 61-88.

Lampert, M. (2001). *Teaching problems and the problems of teaching*. New Haven, CT: Yale University Press.

Lampert, M. (2010). Learning teaching in, from, and for practice: What do we mean? *Journal of Teacher Education, 61*(1-2), 21-34.

Lampert, M., & Ball, D. L. (1998). *Teaching, Multimedia, and Mathematics: Investigations of Real Practice. The Practitioner Inquiry Series*. Teachers College Press,

Lampert, M., Franke, M. L., Kazemi, E., Ghousseini, H., Turrou, A. C., Beasley, H., CUNARD, A., & Crowe, K. (2013). Keeping it complex using rehearsals to support novice teacher learning of ambitious teaching. *Journal of Teacher Education*, *64*(3), 226-243.

Lave, J., & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation*. Cambridge university press.

Leinhardt, G. (1989). Math lessons: A contrast of novice and expert competence. *Journal for Research in Mathematics Education, 20*(1), 52-75.

Leinhardt, G. (2001). Instructional explanations: A commonplace for teaching and location for contrast. In V. Richardson (Ed.), H*andbook for research on teaching (4th Ed.).* Washington, DC: American Educational Research Associates.

Leinhardt, G., & Greeno, J. G. (1986). The cognitive skill of teaching. *Journal of Educational Psychology, 78*(2), 75-95.

Leinhardt, G., & Steele, M.D. (2005). Seeing the complexity of standing to the side: Instructional dialogues. *Cognition and Instruction*, *23*(1), 87-163.

Lewis, J.M. (2007). *Teaching as Invisible Work*. Unpublished doctoral dissertation, University of Michigan, Ann Arbor, MI.

Livingston, C., & Borko, H. (1989). Expert-novice differences in teaching: A cognitive analysis and implications for teacher education. *Journal of Teacher Education*, *40*(4), 36-42.

Lortie, D. C. (1975). *School teacher: A sociological inquiry*. Chicago: University of Chicago Press.

Mancosu, P. (2001). Mathematical explanation: Problems and prospects. *Topoi*,*20*(1), 97-117.

Mariotti, M. A. (2000). Introduction to proof: The mediation of a dynamic software environment. *Educational Studies in Mathematics*, *44*(1-2), 25-53.

Morris, A. K., Hiebert, J., & Spitzer, S. M. (2009). Mathematical knowledge for teaching in planning and evaluating instruction: What can preservice teachers learn?. *Journal for Research in Mathematics Education, 40*(5), 491-529.

National Council of Teachers of Mathematics. (1991). *Professional Standards for Teaching Mathematics*. Reston, VA: The author.

National Council of Teachers of Mathematics. (2000). *Principles and Standards for School Mathematics.* Reston, VA: The author.

National Research Council. (2001). *Adding It Up: Helping Children Learn Mathematics.* Washington, DC: National Academy Press.

Parsons, T. (1938). The role of theory in social research. *American Sociological Review*, *3*(1), 13-20.

Patton, M. Q. (2002). Qualitative evaluation and research methods (3rd ed.). Thousand Oaks, CA: Sage Publications, Inc.

Pedemonte, B. (2007). How can the relationship between argumentation and proof be analysed? *Educational Studies in Mathematics, 66*(2), 23-41.

Perry, M. (2000). Explanations of mathematical concepts in Japanese, Chinese, and U.S. first- and fifth-grade classrooms. *Cognition and Instruction, 18*(2), 181-207.

Pirsig, R. M. (1974). *Zen and the art of motorcycle maintenance: An inquiry into values*. Random House.

Schoenfeld, A. H. (2011). *How we think: A theory of goal-oriented decision making and its educational applications*. Routledge.

Shadish, W.R., Cook, T. D., & Campbell, D. T. (2002). *Experimental and quasi-experimental designs for generalized causal inference*. Wadsworth Cengage learning.

Shavelson, R. J. (1973). What is the basic teaching skill? *Journal of Teacher Education*, *24*(2), 144-151.

Sherin, M. G. (2002). A balancing act: Developing a discourse community in a mathematics classroom. *Journal of Mathematics Teacher Education*, *5*(3), 205-233.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Tesearcher*, 4-14.

Simon, M. A., & Blume, G. W. (1996). Justification in the mathematics classroom: A study of prospective elementary teachers. *The Journal of Mathematical Behavior, 15*(1), 3-31.

Sleep, L. (2009). *Teaching to the mathematical point: Knowing and using mathematics in teaching*. Unpublished doctoral dissertation, University of Michigan, Ann Arbor, MI.

Staples, M, E., Bartlo, J., Thanheiser, E. (2010). Justification as a teaching and learning practice: Its (potential) multifaceted role in middle grades mathematics classrooms. *The Journal of Mathematical Behavior, 31*(4), 447-462.

Stein, M. K., Engle, R. A., Smith, M. S., & Hughes, E. K. (2008). Orchestrating productive mathematical discussions: Five practices for helping teachers move beyond show and tell. *Mathematical Thinking and Learning*, *10*(4), 313-340.

Stein, M. K., Grover, B. W., & Henningsen, M. (1996). Building student capacity for mathematical thinking and reasoning: An analysis of mathematical tasks used in reform classrooms. *American Educational Research Journal, 33*(2), 455-488.

Stylianides, A. J. (2007). Proof and proving in school mathematics. *Journal for Research in Mathematics Education, 38*(3), 289-321.

Stylianides, A. J., & Ball, D. L. (2008). Understanding and describing mathematical knowledge for teaching: Knowledge about proof for engaging students in the activity of proving. *Journal of Mathematics Teacher Education*, *11*(4), 307-332.

Thames, M. H. (2009). *Coordinating mathematical and pedagogical perspectives in practice-based and discipline-grounded approaches to studying mathematical knowledge for teaching (K-8)*. Unpublished doctoral dissertation, University of Michigan, Ann Arbor, MI.

Toulmin, S. E. (2003). *The uses of argument*. Cambridge University Press.

Tyminski, A. M., Land, T. J., Drake, C., Zambak, V. S., & Simpson, A. (2014). Preservice elementary mathematics teachers' emerging ability to write problems to build on children's mathematics. In *Research Trends in Mathematics Teacher Education* (pp. 193-218). Springer International Publishing.

Watson, J. M., & Chick, H. L. (2001). Factors influencing the outcomes of collaborative mathematical problem solving: An introduction. *Mathematical Thinking and Learning, 3*(2-3), 125-173.

Webb, N. M. (1991). Task-related verbal interaction and mathematics learning in small groups. *Journal for Research in Mathematics Education, 22*(5), 366-389.

Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education, 27*(4), 458-477.

Yackel, E., Cobb, P., & Wood, T. (1991). Small-group interactions as a source of learning opportunities in second-grade mathematics. *Journal for Research in Mathematics Education, 22*(5), 390-408

Yackel, E., Cobb, P., & Wood, T. (1999). The interactive constitution of mathematical meaning in one second grade classroom: An illustrative example. *The Journal of Mathematical Behavior, 17*(4), 469-488.